Geometrical Structure of Gauge Theories: Electromagnetism, Gravitation

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Abstract

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The geometric structure of gauge natural theories is investigated. We study especially the Einstein-Yang-Mills theory, an example of gauge natural theory, describing the interaction of gravity with the Yang-Mills field. We consider the Yang-Mills part of the theory with a general Lie group G, at no cost in complications, the choice $G = U(1)$ corresponds to electromagnetism. The global variational functional, defined by the Hilbert-Yang-Mills Lagrangian over a smooth manifold, is investigated within the framework of prolongation theory of principal fiber bundles, and global variational theory on fibered manifolds. The principal Lepage equivalent of this Lagrangian is constructed, and the corresponding infinitesimal first variation formula is obtained. It is shown, in particular, that the Noether currents, associated with isomorphisms of the underlying geometric structures, split naturally to several terms, one of which is exterior derivative of the Komar-Yang-Mills superpotential. Consequences of invariance of the Hilbert-Yang-Mills Lagrangian under isomorphisms of underlying geometric structures such as Noether's conservation laws for global currents are then established. We give also some examples of Komar-Yang-Mills superpotentials corresponding to several solutions of the Einstein equations.

Abstrakt

Geometrická struktura kalibračních teorií: elektromagnetismus, gravitace

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Je zkoumána geometrická struktura přirozeně kalibračních teorií. Studujeme především Einsteinovu-Yangovu-Millsovu teorii, příklad přirozeně kalibrační teorie, popisující interakci gravitace s Yangovým-Millsovým polem. Yangovu-Millsovu část teorie uvažujeme, bez větších komplikací, s obecnou Lieovou grupou G, volba $G = U(1)$ odpovídá elektromagnetismu. Globální variační funkcionál, definovaný Hilbertovým-Yangovým-Millsovým lagrangiánem nad hladkou varietou, je zkoumán pomocí prolongační teorie hlavních fibrovaných prostorů a globální variační teorie na fibrovaných varietách. Je zkonstruován hlavní Lepageův ekvivalent tohoto lagrangiánu a získána odpovídající infinitezimální první variační formule. Zvláště je ukázáno, že Noetherovské proudy, asociované s izomorfismy podkladových geometrických struktur, se přirozeně štěpí na několik členů, jeden z nich je vnější derivace Komarova-Yangova-Millsova superpotenciálu. Pak jsou uvedeny důsledky invariance Hilbertova-Yangova-Millsova lagrangiánu vůči izomorfismům podkladových geometrických struktur, jako Noetherovské zákony zachování pro globální proudy. Předkládáme také příklady Komarova-Yangova-Millsova superpotenciálu pro několik řešení Einsteinových rovnic.

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Contents

Chapter 1

Introduction

The goal of this thesis is to investigate the geometric structure of gauge natural theories. The main part of this work is devoted to the global variational formulation of the Einstein-Yang-Mills theory. We decided to include the proofs in this thesis, in the hope that it will be more readable, comprehensible, and controllable. It can serve also as a brief introductory text to the study of the gauge natural structure of classical field theories.

To get a description of physical and geometrical phenomena, many authors prefer classical coordinate approach, and use classical concepts of variations; global and structure aspects of the theory have often been left quite aside. It seems that one of the main reasons for this consist in indistinguishable possibilities how to replace a basic notion for globalization, the Poincaré-Cartan form in the first order variational calculus, with its suitable generalization for higher order problems.

Our exposition of the subject is based on the prolongation theory of principal fiber bundles due to Kolář ([25, 34] and the references therein) and the general variational theory on fibered manifolds due to Krupka (see e.g. [8, 30, 31]). We are led to a definition of a gauge natural structure of gauge natural field theories. We remark that similar approach is in [16]. We use Lepage forms and Kolář's prolongation theory with the aim to give new, exact exposition on the Hilbert-Yang-Mills functional. We also apply the theory to several known examples. Our results agree with known predictions from theoretical physics.

The trends to extend coordinate understanding of physical laws and phenomena to smooth manifolds have successfully modified many disciplines of mathematics and mathematical physics. First steps, emphasizing the geometric structure of the Einstein-Yang-Mills theory, were made by Bleecker [4]. Fatibene and Francaviglia [16] interpreted the Einstein-Yang-Mills theory by means of a variational principle for sections of fiber bundles; the underlying variational concepts are Lepage forms (Krupka [28, 29]), generalizing the well-known concept of the first order Poincaré-Cartan form.

Our contribution in this thesis consists in the following innovations. We prove the Utiyama-like theorem for the Einstein-Yang-Mills theory by applying the orbit reduction method, a powerful method for computing invariants of group actions. We systematically use the principal prolongation theory of principal fiber bundles; in particular, the prolongation theory gives us a general formula for prolongations of the generators of automorphisms of the underlying structure bundle to configuration bundle. We also give a new direct proof for the splitting of the currents in the Einstein-Yang-Mills theory into three summands, one of which is the exterior derivative of the Komar-Yang-Mills superpotential. We show that the theory allows us to compute the most general expression for the Komar-Yang-Mills superpotential, and we find an explicit expression for several solutions of the Einstein equations.

This work differs conceptually from [16] in several aspects. We prefer differential forms, which describe the underlying global structures of the theory. The main reason for this consists in the fact that the first variation formula contains the exterior derivative operator d. In particular, d is an essential operator, describing the global structure of many variational constructions (see [31]). It should also be pointed out that our equations of motion for the Einstein-Yang-Mills fields differ from the equations derived in [16].

The thesis is organized as follows. Chapter 2 is devoted to main definitions and results of the theory of gauge natural bundles and operators. This geometric background of many physical theories is well suited for the description of their invariance properties like the independence on diffeomorphisms and the gauge transformations. In Chapter 3 we give a survey of the general variational theory and we focus our attention on the concepts needed in the Einstein-Yang-Mills theory. A basic element of the theory is the so called principal Lepage form introduced in [28], a Lepage equivalent of the second order Lagrangian λ, that enjoys similar properties as the first order Poincaré-Cartan form. In Chapter 4 we study the geometric structure of the Einstein-Yang-Mills theory. The gravitational field and the Yang-Mills field are considered together as a section of an appropriate fibered manifold. We introduce the Hilbert-Yang-Mills functional, whose Lagrangian λ is the sum of the Hilbert Lagrangian for a free metric field on a manifold X, and the Yang-Mills Lagrangian for a principal connection field on X. We derive the principal Lepage equivalent of the Hilbert-Yang-Mills Lagrangian and give the corresponding (global, infinitesimal) first variation formula. We analyze the invariance of λ with respect to isomorphisms of underlying geometric structures, the manifold X and a principal G-bundle over X. Further we discuss the first variation formula for induced variations. In Chapter 5, we study some examples. We analyze the Komar-Yang-Mills superpotential for some solutions of the Einstein-Yang-Mills equations: the Levi-Civita-Bertotti-Robinson solution, the Reissner-Nordström solution, the Kerr-Newman solution and the so called embedded Abelian solution (the colored black hole); further we comment on the conserved quantities.

Chapter 2

Gauge Natural Bundles and Operators

A fiber bundle is the generalization of the well known tangent space. Bundles play a very important role in mathematics as well as in physics. For example the jet bundle is the main structure, which appears in the calculus of variations, and gauge natural bundles serve in theoretical physics as configuration spaces. Geometric objects from differential geometry and matter fields from physics can be considered as sections of some bundles. In this chapter we introduce jets, gauge natural bundles and their operators. For the notation and terminology in this chapter we refer to [25, 34].

2.1 Jets and Gauge Natural Bundles

Two curves $\gamma, \delta : \mathbb{R} \to X$ in a manifold X have **r-th contact at zero** and we write $\gamma \sim_r \delta$, if for every smooth function ϕ on X the difference $\phi \circ \gamma - \phi \circ \delta$ vanishes to r-th order at $0 \in \mathbb{R}$, i.e. all derivatives up to order r of the difference vanish at $0 \in \mathbb{R}$ ($\gamma \sim_0 \delta$ means $\gamma(0) = \delta(0)$). The relation $\sim_r$ is obviously an equivalence relation. Two maps $f, g : X \to Y$ between two manifolds X and Y are **tangent to order r at $x \in X$**, if for every curve $\gamma : \mathbb{R} \to X$ with $\gamma(0) = x$ holds $f \circ \gamma \sim_r g \circ \gamma$. This is an equivalence relation too and the equivalence class whose representative is a map f is called **r-jet of f at x** (or simply a **jet**) and is denoted by $J^r_x f$. The set of all r-jets of X into Y is denoted by $J^r(X, Y)$. The map $J^r_x f \mapsto x$ sending a jet to its **source** is called the **source projection** and is denoted by α. The map $J^r_x f \mapsto f(x)$

sending a jet to its **target** is called the **target projection** and is denoted by β. We denote by $\pi^{r,s}$, $0 \leq s \leq r$ the **canonical projection** $J^r_x f \mapsto J^s_x f$ of r-jets into s-jets. We write $J^r_x(X,Y)$ or $J^r(X,Y)_y$ for the set of all r-jets of X into Y with source $x \in X$ or target $y \in Y$, respectively, and $J^r_x(X,Y)_y = J^r_x(X,Y) \cap J^r(X,Y)_y$. The map $J^r f : X \to J^r(X,Y)$ given by $J^r f(x) = J^r_x f$ is called the r**-jet prolongation** of $f : X \to Y$.

The following theorem can serve as an equivalent definition of a jet (as in [34] or [32]).

Theorem 2.1. *Two maps $f, g : X \to Y$ satisfy $J^r_x f = J^r_x g$ iff there exists a chart (U, φ) at x and a chart (V, ψ) at $f(x)$ such that*

$$D^k(\psi \circ f \circ \varphi^{-1})(\varphi(x)) = D^k(\psi \circ g \circ \varphi^{-1})(\varphi(x)) \tag{2.1}$$

holds for each k, $0 \leq k \leq r$.

Proof: First we remark that in components, $(f^p) = (y^p \circ f \circ \varphi^{-1}) = \psi \circ f \circ \varphi^{-1}$, $(g^p) = (y^p \circ g \circ \varphi^{-1}) = \psi \circ g \circ \varphi^{-1}$, $1 \leq p \leq \dim(Y)$ the condition (2.1) for f, g to be tangent to order r at $x \in X$ is equivalent to the following condition: $f(x) = g(x)$ and

$$D_{i_1} D_{i_2} \dots D_{i_k} f^p(\varphi(x)) = D_{i_1} D_{i_2} \dots D_{i_k} g^p(\varphi(x))$$

for each k, $1 \leq k \leq r$, where $1 \leq i_1, i_2, \dots, i_k \leq n = \dim(X)$, i.e. all the partial derivatives up to order r of the components f^p and g^p coincide at $\varphi(x)$. In fact, its enough to recall the identity from differential calculus for a map $\tilde{f}$ from some open set in $\mathbb{R}^n$ into a Banach space having k-th derivation (see [12]); we have with $t_i = (\xi_{ij})$ $(1 \leq i \leq k,\ 1 \leq j \leq n)$

$$D^k \tilde{f}(\tilde{x}) \cdot (t_1, \dots, t_k) = \sum_{(j_1, j_2, \dots, j_k)} D_{j_1} D_{j_2} \dots D_{j_k} \tilde{f}(\tilde{x}) \xi_{1j_1} \xi_{2j_2} \dots \xi_{kj_k},$$

where we sum over all n^k possibilities.

Now we deduce that two curves $\gamma, \delta : \mathbb{R} \to Y$ satisfy $\gamma \sim_r \delta$ iff

$$\frac{d^k(y^p \circ \gamma)(0)}{dt^k} = \frac{d^k(y^p \circ \delta)(0)}{dt^k} \tag{2.2}$$

for each k, $1 \leq k \leq r$, and for all coordinate functions y^p. In fact, $\gamma \sim_r \delta$ implies that $y^p \circ \gamma - y^p \circ \delta$ vanishes to order r, and so (2.2) holds. Conversely, we first recall the **higher order chain rule**. We use the shorthand notation. For a set of positive integers $I = \{i_1, i_2, \dots, i_k\}$, $1 \leq i_1, i_2, \dots, i_k \leq n$, we denote by $D_I = D_{i_1} D_{i_2} \dots D_{i_k}$ the partial derivative. Let $\tilde{U} \subset \mathbb{R}^n$ and $\tilde{V} \subset \mathbb{R}^m$ be open sets, let $\tilde{f} : \tilde{V} \to \mathbb{R}$ be a smooth function, and let $\tilde{g} = (\tilde{g}^\sigma)$, $1 \leq \sigma \leq m$, be a smooth mapping of $\tilde{U}$ into $\tilde{V}$. Then we have

$$\begin{aligned} &D_{i_s} \dots D_{i_2} D_{i_1} (\tilde{f} \circ \tilde{g})(t) \\ &= \sum_{k=1}^{s} \sum_{(I_1, I_2, \dots, I_k)} D_{\sigma_k} \dots D_{\sigma_2} D_{\sigma_1} \tilde{f}(\tilde{g}(t)) D_{I_k} \tilde{g}^{\sigma_k}(t) \dots D_{I_2} \tilde{g}^{\sigma_2}(t) D_{I_1} \tilde{g}^{\sigma_1}(t), \end{aligned}$$

where the second sum is understood to be extended to all partitions $(I_1, I_2, \dots, I_k)$ of the set $\{i_1, i_2, \dots, i_s\}$. This can be proved by induction (see [32]). Given a function $\tilde{\phi}$ on Y, we find by the higher order chain rule that all partial derivatives (and so all derivatives) up to order r of $\tilde{\phi} \circ \gamma$ at zero depend only on the partial derivatives up to order r of $\tilde{\phi}$ at $(y^p \circ \gamma)(0)$ and on the derivatives $\frac{d^k(y^p \circ \gamma)(0)}{dt^k}$, which occur in Equation (2.2). Hence $\tilde{\phi} \circ \gamma - \tilde{\phi} \circ \delta$ vanishes to order r at 0 and Equation (2.2) really implies $\gamma \sim_r \delta$.

If we suppose that the partial derivatives up to the order r of f^p and g^p coincide at $\varphi(x)$, then the higher order chain rule implies $f \circ \gamma \sim_r g \circ \gamma$ by the previous paragraph. Thus we get $J^r_x f = J^r_x g$. Conversely, if we suppose that $J^r_x f = J^r_x g$ holds, then using the curves of the form $\varphi \circ \gamma(t) = at$ for arbitrary $a \in \mathbb{R}^n$ we get from $f \circ \gamma \sim_r g \circ \gamma$ in the coordinates $\sum_{|i|=k} (D_i f^p(0)) a^i = \sum_{|i|=k} (D_i g^p(0)) a^i$ (with $0 \le k \le r$), where $i = (i_1, \dots, i_n)$ is an n-tuple of non-negative integers, so called **multiindex** of range n, $a^i = (a^1)^{i_1} \dots (a^n)^{i_n}$ for $a = (a^1, \dots, a^n) \in \mathbb{R}^n$, $|i| = i_1 + \dots + i_n$, $D_i \tilde{f} = \frac{\partial^{|i|} \tilde{f}}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$ for a function $\tilde{f}$ from some open subset of $\mathbb{R}^n$ into $\mathbb{R}$. Since a is arbitrary, we get that all the partial derivatives up to order r of the components f^p and g^p coincide at $\varphi(x)$.

$\diamondsuit$

We can define the composition $J^r_y g \circ J^r_x f \in J^r_x(X, Z)_z$ of r-jets $J^r_x f \in J^r_x(X, Y)_y$ and $J^r_y g \in J^r_y(Y, Z)_z$ by $J^r_y g \circ J^r_x f = J^r_x(g \circ f)$. We show that this is well defined. If we suppose that $J^r_x f = J^r_x \bar{f}$ and $J^r_x g = J^r_x \bar{g}$, $f(x) = y = \bar{f}(x)$, then we can write for the other representatives $J^r_x(g \circ f) = J^r_x(\bar{g} \circ \bar{f})$. In fact, $J^r_x f = J^r_x \bar{f}$ means that $f \circ \gamma \sim_r \bar{f} \circ \gamma$ holds for every curve $\gamma : \mathbb{R} \to X$ with $\gamma(0) = x$. From this we immediately get $\bar{g} \circ f \circ \gamma \sim_r \bar{g} \circ \bar{f} \circ \gamma$, but $J^r_x g = J^r_x \bar{g}$ yields $g \circ f \circ \gamma \sim_r \bar{g} \circ f \circ \gamma$, thus $g \circ f \circ \gamma \sim_r \bar{g} \circ \bar{f} \circ \gamma$ holds for every curve $\gamma : \mathbb{R} \to X$ with $\gamma(0) = x$. Therefore we really get $J^r_x(g \circ f) = J^r_x(\bar{g} \circ \bar{f})$ and the composition of r-jets is well defined. An r-jet $A \in J^r_x(X, Y)_y$ is called **regular**, if there exists an r-jet $B \in J^r_y(Y, X)_x$ such that $B \circ A = J^r_x \mathrm{id}_X$ and A is called **invertible**, if there exists an r-jet $A^{-1} \in J^r_y(Y, X)_x$ such that $A^{-1} \circ A = J^r_x \mathrm{id}_X$ and $A \circ A^{-1} = J^r_y \mathrm{id}_Y$. We denote by $\mathrm{inv}\, J^r(X, Y)$ the set of invertible r-jets of X into Y. It is not difficult to see (the proof is in [32]) that an r-jet $X \in J^r_x(X, Y)_y$ is regular if and only if each of its representatives is an immersion at the point x and it is invertible if and only if each of its representatives is a local diffeomorphism at x. Let $f : X \to \bar{X}$ be a local diffeomorphism, and $g : Y \to \bar{Y}$ be a smooth map. Then there exists an induced map $J^r(f, g) : J^r(X, Y) \to J^r(\bar{X}, \bar{Y})$ defined by $J^r(f, g)(A) = (J^r_{\beta(A)} g) \circ A \circ (J^r_{\alpha(A)} f)^{-1}$.

A triple (Y, π, X), where $\pi : Y \to X$ is a surjective submersion, is called a **fibered manifold**. Y is called the **total space**, X is called the **base space**, π is called the **projection**. Since π is a surjective submersion, it is transversal over $x \in X$, therefore $\pi^{-1}(x)$ is a submanifold of Y (see [36]), which is called the **fiber** of Y over x and we sometimes write Y_x instead of $\pi^{-1}(x)$. A **morphism of fibered manifolds** (Y, π, X) and $(\bar{Y}, \bar{\pi}, \bar{X})$ is a smooth map $f : Y \to \bar{Y}$ transforming each fiber of Y into a fiber of $\bar{Y}$, i.e. there exists a map $f_0 : X \to \bar{X}$

such that the following diagram is commutative:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"'] & \bar{Y} \arrow[d, "\bar{\pi}"] \\ X \arrow[r, "f_0"'] & \bar{X} \end{tikzcd}$$

From the universal property of surjective submersion we get that f_0 is smooth. We denote the category of all fibered manifolds and their morphisms by $\mathcal{FM}$ and by ΓY the set of smooth sections of Y. We denote by B the **base functor** from the category of fibered manifolds into the category of manifolds $B : \mathcal{FM} \to \mathcal{M}f$, which sends every fiber manifold (Y, π, X) down to its base X and every fibered manifold morphism f to f_0. We denote by $\mathcal{FM}_n$ the subcategory of fibered manifolds with n-dimensional bases and morphisms of fibered manifolds with local diffeomorphisms as base maps. We denote by $\mathcal{M}f_n$ the subcategory of $\mathcal{M}f$- the category of manifolds and smooth mappings, where we consider n-dimensional manifolds and local diffeomorphism.

Theorem 2.2. *Let X and Y be smooth manifolds. There exists an induced structure of smooth manifold on $J^r(X, Y)$ such that r-jet projections are smooth surjective submersions, the composition of r-jets is smooth and J^r is a functor $\mathcal{M}f_n \times \mathcal{M}f \to \mathcal{FM}$.*

Proof: Let (U, φ), $\varphi = (x^i)$ be a chart on X and let (V, ψ), $\psi = (y^p)$ be a chart on Y. We set $W = (\pi^{r,0})^{-1}(U \times V)$ and put for each $J^r_x f \in W$

$$\begin{aligned} \chi(J^r_x f) &= (x^i \circ \alpha, y^p \circ \beta, y^p_{i_1}, y^p_{i_1 i_2}, \dots, y^p_{i_1 i_2 \dots i_r})(J^r_x f), \\ y^p_{i_1 i_2 \dots i_k}(J^r_x f) &= D_{i_1} D_{i_2} \dots D_{i_k}(y^p \circ f \circ \varphi^{-1})(\varphi(x)), \end{aligned}$$

where $1 \le k \le r$, $1 \le p \le m = \dim(Y)$ and $1 \le i_1 \le i_2 \le \dots \le i_r \le n = \dim(X)$. We show that $\chi : W \to \varphi(U) \times \psi(V) \times \mathbb{R}^N$ is a bijection, where using the combination with repetition we see that

$$N = m\left(\binom{n}{1} + \binom{n+1}{2} + \dots + \binom{n+r-1}{r}\right) = m\left(\binom{n+r}{n} - 1\right).$$

It follows immediately from Theorem 2.1 (see the remark at the beginning of its proof) that χ is injective. To show that it is surjective, choose a point $(x_0 = (x^i_0), y_0 = (y^p_0), P^p_{i_1}, P^p_{i_1 i_2}, \dots, P^p_{i_1 i_2 \dots i_r}) \in \varphi(U) \times \psi(V) \times \mathbb{R}^N$ and define $P^p_{j_1 j_2 \dots j_k} = P^p_{i_1 i_2 \dots i_k}$ whenever $(j_1, j_2, \dots, j_k)$ is a permutation of $(i_1, i_2, \dots, i_k)$. Now we can define a map $\tilde{f} = (\tilde{f}^p) : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\begin{aligned} \tilde{f}^p(x^1, \dots, x^n) = y^p_0 + P^p_{j_1}(x^{j_1} - x^{j_1}_0) + \frac{1}{2!} P^p_{j_1 j_2}(x^{j_1} - x^{j_1}_0)(x^{j_2} - x^{j_2}_0) \dots \\ + \frac{1}{r!} P^p_{j_1 j_2 \dots j_r}(x^{j_1} - x^{j_1}_0)(x^{j_2} - x^{j_2}_0) \dots (x^{j_r} - x^{j_r}_0). \end{aligned}$$

Putting $f = \psi^{-1} \circ \tilde{f} \circ \varphi$, $x = \varphi^{-1}(x_0)$ we obtain a smooth map such that $\chi(J^r_x f) = (x_0, y_0, P^p_{i_1}, P^p_{i_1 i_2}, \dots, P^p_{i_1 i_2 \dots i_r})$. Therefore χ is really a bijection. Using the higher order chain rule we see that the chart changings are smooth maps, further we see that the canonical projections look locally like a projections and so they are surjective submersions, specially this defines the structure of a smooth fibered manifold on $\pi^{r,0} : J^r(X,Y) \to X \times Y$. The chart (W, χ) on the manifold $J^r(X,Y)$ is said to be **associated** with the charts (U, φ) and (V, ψ). Using the higher order chain rule again, we see that the coordinates of a jet $J^r_y g \circ J^r_x f$ depend polynomially on the coordinates of the jets $J^r_y g$, $J^r_x f$, therefore the composition of jets is smooth. Finally, since the composition of jets is associative, we obtain a functor $J^r : \mathcal{M}f_n \times \mathcal{M}f \to \mathcal{FM}$, which sends a pair of manifolds (X,Y) to a fibered manifold $\pi^{r,0} : J^r(X,Y) \to X \times Y$ and a pair of morphism (f,g) to an induced map $J^r(f,g)$, which is obviously a $\mathcal{FM}$-morphism over (f,g).

$\diamondsuit$

Example 2.1. If we define L^r_n as the set of all invertible elements of $J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$ with an operation on it given by the composition of jets $L^r_n \times L^r_n \to L^r_n$, $(A, B) \mapsto A \circ B$, then L^r_n is a Lie group called the r-th **differential group** or the r**-th jet group in dimension** n. We can introduce the **canonical coordinates on** $J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$, and so on L^r_n as previously by

$$a^p_{i_1 i_2 \dots i_k}(J^r_0 f) = D_{i_1} D_{i_2} \dots D_{i_k} f^p(0),$$

where $f = (f^p) : \mathbb{R}^n \to \mathbb{R}^n$, $1 \le k \le r$, $1 \le i_1 \le i_2 \le \dots \le i_k \le n$. In these coordinates we have $L^r_n = \{J^r_0 f \in J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0 : \det a^p_i(J^r_0 f) \neq 0\}$. Since the mapping $\det \circ a^p_i : J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0 \to \mathbb{R}$ is continuous, $J^r_0 f \in L^r_n$ has a neighborhood on which this function is nonzero. We can unify all such neighborhoods to prove that L^r_n is open in $J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$. This defines the structure of a smooth manifold on L^r_n. Our operation is associative, the r-jet $J^r_0 \mathrm{id}_{\mathbb{R}^n}$ is the unit, every r-jet $J^r_0 f \in L^r_n$ has an inverse $J^r_0 f|^{-1}$, where $f|$ denotes a restriction of f on some neighborhood on which f is a diffeomorphism. Thus L^r_n is a group. Theorem 2.2 implies that our operation is smooth, therefore L^r_n is really a Lie group. We remark that $\dim L^r_n = n\left(\binom{n+r}{n} - 1\right)$ and L^1_n can be identified with the general linear group $GL(n, \mathbb{R})$. Further the higher order chain rule (using the same notation) implies that the group operation can be written in canonical coordinates in the form

$$a^p_{i_1 i_2 \dots i_s}(J^r_0 \alpha \circ J^r_0 \beta) = \sum_{k=1}^{s} \sum_{(I_1, I_2, \dots, I_k)} a^p_{j_1 j_2 \dots j_k}(J^r_0 \alpha) a^{j_1}_{I_1}(J^r_0 \beta) a^{j_2}_{I_2}(J^r_0 \beta) \dots a^{j_k}_{I_k}(J^r_0 \beta).$$

Example 2.2. Similarly as in Theorem 2.2 (see [32] or [25]) it can be proved that $T^r_k X = J^r_0(\mathbb{R}^k, X) \to X$ is a fiber bundle and $T^r_k : \mathcal{M}f \to \mathcal{FM}$ is a functor, which is on morphisms given by $T^r_k f(J^r_0 g) = J^r_0(f \circ g)$. The elements of the manifold $T^r_k X$ are said to be the k**-dimensional velocities of order** r on X, in short (k, r)**-velocities**.

For every Lie group G with the multiplication $\mu : G\times G \to G$, $T^r_k G$ is also a Lie group with the multiplication $T^r_k\mu : T^r_k G\times T^r_k G\to T^r_k G$. We use the fact that T^r_k preserves products, i.e. $T^r_k G\times T^r_k G \cong T^r_k(G\times G)$ with the identification given by $(J^r_0 f, J^r_0 g) \mapsto J^r_0(f, g)$. It is well defined, it is enough to use Theorem 2.1 and recall the following fact from differential calculus (see [12] or [36]): Let U be open in the Banach space E and let $f_i : U \to F_i$ $(i = 1,\dots,m)$ be continuous maps into the Banach spaces F_i. Let $f = (f_1,\dots, f_m)$ be the map of U into the product of the F_i. Then f is of class C^k iff each f_i is of class C^k, and in that case $D^k f = (D^k f_1,\dots, D^k f_m)$.

Analogously we define the space of all (k, r)-**covelocities** on X by $T^{r*}_k X = J^r(X,\mathbb{R}^k)_0$. T^{r*}_k is a functor from $\mathcal{M}f_n$, on morphisms given by $T^{r*}_k f(J^r_x g) = J^r_{f(x)}(g\circ f^{-1})$ (where f^{-1} is constructed locally as in Example 2.1), into L^r_n- bundles ($\dim(X) = n$). We show that $\alpha : T^{r*}_k X\to X$ is really a L^r_n-bundle. We observe that $\mathbb{R}^k$ induce the structure of a vector space in each fiber $\alpha^{-1}(x)$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be an atlas on X. We can define the local trivializations $\phi_\beta : \alpha^{-1}(U_\beta)\to U_\beta \times J^r_0(\mathbb{R}^n,\mathbb{R}^k)_0$ by $\phi_\beta(J^r_x f) = (\alpha(J^r_x f), J^r_0(f\circ\varphi^{-1}_\beta \circ t_{\varphi_\beta(x)}))$, where we use the translation on $\mathbb{R}^n$ given by $t_a(b) = b+a$ for $a, b\in\mathbb{R}^n$. For the inverse map to ϕ_β we have $\phi^{-1}_\beta(x, J^r_0 g) = J^r_x(g\circ t_{-\varphi_\beta(x)}\circ\varphi_\beta)$. Thus we get

$$\begin{aligned}\phi_\alpha\circ\phi^{-1}_\beta(x, J^r_0 g) &= (x, J^r_0(g\circ t_{-\varphi_\beta(x)}\circ\varphi_\beta\circ\varphi^{-1}_\alpha\circ t_{\varphi_\alpha(x)}))\\ &= (x, J^r_0 g\circ J^r_0(t_{-\varphi_\beta(x)}\circ\varphi_\beta\circ\varphi^{-1}_\alpha\circ t_{\varphi_\alpha(x)})).\end{aligned}$$

We define the left action $L^r_n \times J^r_0(\mathbb{R}^n,\mathbb{R}^k)_0\to J^r_0(\mathbb{R}^n,\mathbb{R}^k)_0$ by $l(J^r_0 h, y) = y\circ (J^r_0 h)^{-1}$. The left action as a composition of smooth maps is obviously smooth. Further we set $\phi_{\alpha\beta}(x) = J^r_0(t_{-\varphi_\alpha(x)}\circ\varphi_\alpha\circ\varphi^{-1}_\beta\circ t_{\varphi_\beta(x)})$, these $\phi_{\alpha\beta}$ are smooth. Hence we get $l(\phi_{\alpha\beta}(x), J^r_0 g) = \mathrm{pr}_2\circ\phi_\alpha\circ\phi^{-1}_\beta(x, J^r_0 g)$ and $\{\phi_{\alpha\beta}\}_{\alpha,\beta\in I}$ is a cocycle of transition functions for the L^r_n-bundle $\alpha : T^{r*}_k X\to X$.

Example 2.3. The set $F^r X$ of all r-jets with source 0 of the local diffeomorphism of $\mathbb{R}^n$ into X is called the **frame bundle of order** r of X. So an r**-frame** at $x\in X$ is an invertible (n, r)-velocity at a point x. We show that $F^r X$ is a principal fiber bundle with structure group L^r_n. The r-th differential group L^r_n acts smoothly on $F^r X$ on the right by jet composition, i.e. we have the right action $r : F^r X\times L^r_n\to F^r X, (u, A)\mapsto u\circ A$. Since u is an invertible jet, we can act by u^{-1} on the equation $u\circ A = u\circ B$ with $u\in F^r X$ and $A, B\in L^r_n$, then $A = B$ and the right action is free. As in Example 2.1 we can prove that $F^r X$ is open in $T^r_n X$, which defines a structure of fiber manifold (bundle) on $\beta : F^r X\to X$. Further for every $J^r_0\phi, J^r_0\psi\in\beta^{-1}(x)$, $x = \beta(u)$ in the same fiber of $\beta : F^r X\to X$ there is a unique element $J^r_0(\phi^{-1}\circ\psi)\in L^r_n$ satisfying $(J^r_0\phi)\circ(J^r_0(\phi^{-1}\circ\psi)) = J^r_0\psi$, thus r is transitive on fibers, hence we get $\beta^{-1}(x)\subset \mathrm{orb}(u)$. Since we have $\beta(u) = \beta(u\circ A)$, we obtain $\beta^{-1}(x)\supset\mathrm{orb}(u)$. Thus $\beta^{-1}(x) = \mathrm{orb}(u)$, i.e. the orbits of the right action are exactly the fibers $\beta^{-1}(x)$ of $F^r X$. Therefore we can apply the following Theorem (see [25]) to prove that $(F^r X, \beta, X, L^r_n)$ is really a principal bundle:

Theorem 2.3. *Let $p : P\to X$ be a fibered manifold, and let G be a Lie group* *which acts freely on P from the right such that the orbits of the action are exactly the fibers $p^{-1}(x)$ of P. Then (P, p, X, G) is a principal fiber bundle.*

Every local diffeomorphism $f : X \to Y$ induces a map $F^r f : F^r X \to F^r Y$ by $F^r f(J_0^r \phi) = J_0^r(f \circ \phi)$. We denote the category of principal G-bundles and their homomorphisms by $\mathcal{PB}(G)$. Since for $J_0^r \alpha \in L_n^r$ we have

$$F^r f(J_0^r \phi \circ J_0^r \alpha) = J_0^r(f \circ \phi \circ \alpha) = J_0^r(f \circ \phi) \circ J_0^r(\alpha) = F^r f(J_0^r \phi) \circ J_0^r(\alpha),$$

$F^r f$ is a smooth L_n^r-equivariant mapping and $F^r : \mathcal{M}f_n \to \mathcal{PB}(L_n^r)$ is a functor.

Example 2.4. Let $\pi : Y \to X$ be a fibered manifold, $\dim X = n$, $\dim Y = n+m$. The set $J^r Y$ of all r-jets of the local sections of Y will be called the **r-jet prolongation** of Y. We see that an element $v \in J_x^r(X, Y)$ belongs to $J^r Y$ if and only if $(J_{\beta(v)}^r \pi) \circ v = J_x^r \mathrm{id}_X$. If we use the associated chart (W, χ) (as in the proof of Theorem 2.2), then we see that for $J_x^r s \in J^r Y \cap W$, $1 \le i \le n$ and $1 \le j_1 \le j_2 \le \cdots \le j_r \le n$ we have

$$y_j^i(J_x^r s) = \delta_j^i,\ y_{j_1 j_2}^i(J_x^r s) = 0, \ldots,\ y_{j_1 j_2 \ldots j_r}^i(J_x^r s) = 0,$$

because the local section s satisfies $y^i \circ s = x^i$. Therefore $J^r Y$ is a submanifold of $J^r(X, Y)$. If there exist a fiber chart (V, ψ), $\psi = (x^i, y^p)$ on Y, then for any multiindex j, $0 \le |j| \le r$ of range n $((\pi^{r,0})^{-1}(V), (x^i, y_j^p))$ is a fiber chart on $J^r Y$, where we denote by the same letter the restriction $\pi^{r,0} : J^r Y \to J^0 Y$ of the canonical projection. For every section s of $\pi : Y \to X$, the r-jet prolongation $J^r s$ of s is a section of $\alpha : J^r Y \to X$.

Let $\bar\pi : \bar Y \to \bar X$ be another fibered manifold and $f : Y \to \bar Y$ be an $\mathcal{FM}$-morphism with the property that the base map $f_0 : X \to \bar X$ is a local diffeomorphism. Then the induced map constructed before Theorem 2.2 $J^r(f_0, f) : J^r(X, Y) \to J^r(\bar X, \bar Y)$, i.e. $J^r(f_0, f)(J_x^r s) = J_{f_0(x)}^r(f \circ s \circ f_0^{-1})$, transforms $J^r Y$ into $J^r \bar Y$. In fact, $v = J_x^r s \in J^r Y$, is characterized by $(J_{\beta(v)}^r \pi) \circ v = J_x^r \mathrm{id}_X$ and since $\mathcal{FM}$-morphism f satisfies $\bar\pi \circ f = f_0 \circ \pi$, we get

$$\begin{aligned}
(J^r_{\beta(J^r(f_0,f)(J_x^r s))}\bar\pi) \circ J^r(f_0, f)(J_x^r s) &= (J^r_{f\circ\beta(v)}\bar\pi) \circ (J^r_{\beta(v)} f) \circ v \circ (J^r_{f_0(x)} f_0^{-1}) \\
= (J^r_{\pi\circ\beta(v)} f_0) \circ (J^r_{\beta(v)}\pi) \circ v \circ (J^r_{f_0(x)} f_0^{-1}) &= (J_x^r f_0) \circ (J_x^r \mathrm{id}_X) \circ (J^r_{f_0(x)} f_0^{-1}) \\
&= J^r_{f_0(x)} \mathrm{id}_{\bar X},
\end{aligned}$$

thus we indeed have $J^r(f_0, f)(J_x^r s) \in J^r \bar Y$. We denote the restricted map by $J^r f : J^r Y \to J^r \bar Y$ and it is called the **r-jet prolongation** of f and we denote the corresponding functor by the same symbol $J^r : \mathcal{FM}_n \to \mathcal{FM}$ as the bifunctor J^r before.

Example 2.5. We can consider all at once. Let (P, p, X, G) be a principal bundle. For $s \ge r$ we can define the **principal prolongation of order** (s, r) **of Lie group** G as the semidirect product of Lie groups $W_n^{s,r} G = L_n^s \rtimes T_n^r G$ with respect to the right action $r : T_n^r G \times L_n^s \to T_n^r G$ given by the composition of jets $r(J_0^r a, J_0^s \alpha) = J_0^r a \circ \pi^{s,r}(J_0^s \alpha)$. The multiplication in a semidirect product $H \rtimes K$

of groups H and K with respect to a right action r of H on K is given by $(h_1, k_1)(h_2, k_2) = (h_1 h_2, r_{h_2}(k_1)k_2)$.

We recall that in any category, the **fiber product** or **pullback** of two morphisms $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ over X consist of an object $Y_1 \times_X Y_2$ and two morphisms $p_1 : Y_1 \times_X Y_2 \to Y_1$ and $p_2 : Y_1 \times_X Y_2 \to Y_2$ such that $f_1 \circ p_1 = f_2 \circ p_2$, and satisfying the universal mapping property: Given an object S and two morphisms $g_1 : S \to X$ and $g_2 : S \to Y$ such that $f_1 \circ g_1 = f_2 \circ g_2$, there exists a unique morphism $g : S \to Y_1 \times_X Y_2$ making the following diagram commutative:

$$\begin{tikzcd}
S \arrow[dr, "g"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& Y_1 \times_X Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & Y_2 \arrow[d, "f_2"'] \\
& Y_1 \arrow[r, "f_1"] & X
\end{tikzcd}$$

We sometimes say that p_1 is the pullback of f_2 by f_1 and also write it as $f_1^*(f_2)$ and similarly we write $Y_1 \times_X Y_2$ as $f_1^*(Y_2)$. So the pullback $(Y_1 \times_X Y_2, p_1, p_2)$ is not determined uniquely but only up to a "unique isomorphism which makes everything commute". If $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are transversal morphisms in the category of manifolds, then $Y_1 \times_X Y_2 = (f_1 \times f_2)^{-1}(\Delta_X)(= \{(y_1, y_2) \in Y_1 \times Y_2 : f(y_1) = g(y_2)\})$ (Δ_X is the diagonal of $X \times X$) together with the morphisms into Y_1 and Y_2 obtained from the projections, is a fiber product of f_1 and f_2 over X (see [36]).

Now $(W^{s,r}P, X, \tilde{p}, W_n^{s,r}G) = (F^s X \times_X J^r P, X, \tilde{p}, L_n^s \rtimes T_n^r G)$ is called the **gauge natural prolongation of order** (s, r) **of the principal bundle** P or (s, r)**-th principal prolongation of the principal bundle** P . Here the projection $\tilde{p}(J_0^s \epsilon, J_x^r \sigma) = x$ is a surjective submersion. We have the free right action of $W_n^{s,r}G$ on $W^{s,r}P$ given by $(J_0^s \epsilon, J_x^r \sigma) \cdot (J_0^s \alpha, J_0^r a) = (J_0^s(\epsilon \circ \alpha), J_x^r(\sigma \cdot (a \circ \alpha^{-1} \circ \epsilon^{-1})))$, where $\cdot$ on the right hand side denotes the right action of G on P. Indeed, we can write the right action of $W_n^{s,r}G$ on $W^{s,r}P$ shortly in the form $(u, v) \cdot (A, B) = (u \circ A, v \cdot (B \circ \pi^{s,r}(A^{-1} \circ u^{-1})))$, $u \in F^s X$, $v \in J^r P$, $A \in L_n^s$, $B \in T_n^r G$ and now $\cdot$ on the right hand side is the induced map from the right action ρ of G on P given by

$$J^r P \times_X J^r(X, G) \to J^r P, \; (J_x^r \sigma, J_x^r s) \mapsto J_x^r \sigma \cdot J_x^r s = J_x^r(\rho \circ (\sigma, s)). \qquad (2.3)$$

We see that $(J_x^r \sigma \cdot J_x^r s) \cdot J_x^r t = J_x^r \sigma \cdot J_x^r(\mu \circ (s, t))$, where μ is the multiplication in G, we can write this equation shortly in the form $(v \cdot S) \cdot T = v \cdot (ST)$,

$v = J^r_x\sigma \in J^rP$, $S = J^r_x s, T = J^r_x t \in J^r(X,G)$. The computation

$$\begin{aligned}(u,v)\cdot((A,B)(C,D)) &= (u,v)\cdot(A\circ C, r_C(B)D)\\ &= (u\circ A\circ C, v\cdot(r_C(B)D\circ\pi^{s,r}((A\circ C)^{-1}\circ u^{-1})))\\ &= (u\circ A\circ C, v\cdot(((B\circ\pi^{s,r}(C))D)\circ\pi^{s,r}(C^{-1}\circ A^{-1}\circ u^{-1})))\\ &= (u\circ A\circ C, v\cdot((B\circ\pi^{s,r}(A^{-1}\circ u^{-1}))(D\circ\pi^{s,r}(C^{-1}\circ A^{-1}\circ u^{-1}))))\\ &= (u\circ A\circ C,(v\cdot(B\circ\pi^{s,r}(A^{-1}\circ u^{-1})))\cdot(D\circ\pi^{s,r}(C^{-1}\circ(u\circ A)^{-1})))\\ &= (u\circ A, v\cdot(B\circ\pi^{s,r}(A^{-1}\circ u^{-1})))\cdot(C,D) = ((u,v)\cdot(A,B))\cdot(C,D)\end{aligned}$$

shows that the action of $W^{s,r}_nG$ on $W^{s,r}P$ is really the right action. If we suppose that $(u,v)\cdot(A,B) = (u,v)$, i.e. $(u\circ A, v\cdot(B\circ\pi^{s,r}(A^{-1}\circ u^{-1}))) = (u,v)$, then we get $A = J^s_0\mathrm{id}_{\mathbb{R}^n}$, because $u \in F^sX$ is invertible. If we write $v = J^r_x\sigma$, $B = J^r_0 b$, $u = J^s_0\epsilon$, then our assumption implies

$$J^r_x(\sigma\cdot(b\circ\epsilon^{-1})) = J^r_x\sigma. \tag{2.4}$$

Let U be a neighborhood of the point $x \in X$ and $\psi : p^{-1}(U)\to U\times G$ a diffeomorphism such that $\psi(y\cdot g) = \psi(y)\cdot g$ for all $y\in p^{-1}(U)$ and $g\in G$, and $\mathrm{pr}_1\circ\psi = p$. Such a diffeomorphism exists, because P is a principal bundle. Then we have

$$\psi(\sigma(x)\cdot(b\circ\epsilon^{-1}(x))) = \psi(\sigma(x))\cdot(b\circ\epsilon^{-1}(x)) = (x,\mu(\mathrm{pr}_2\circ\psi(\sigma(x)), b\circ\epsilon^{-1}(x)))$$

and so we obtain from (2.4) by applying $J^r_{\sigma(x)}(\mathrm{pr}_2\circ\psi)$ from the left and $J^r_0\epsilon$ on the right

$$J^r_0(\mathrm{pr}_2\circ\psi\circ\sigma\circ\epsilon) = J^r_0(\mu\circ(\mathrm{pr}_2\circ\psi\circ\sigma\circ\epsilon, b)) = T^r_n\mu(J^r_0(\mathrm{pr}_2\circ\psi\circ\sigma\circ\epsilon), J^r_0 b),$$

which is the multiplication in the group T^r_nG. Therefore B is the unit in T^r_nG and the right action of $W^{s,r}_nG$ on $W^{s,r}P$ is in fact free. Moreover this right action is transitive on fibers too. Indeed for $(J^s_0\epsilon, J^r_x\sigma), (J^s_0\tilde\epsilon, J^r_x\tilde\sigma) \in \tilde p^{-1}(x)$ there exists $(J^s_0\alpha, J^r_0 a)\in W^{s,r}_nG$ such that $(J^s_0\tilde\epsilon, J^r_x\tilde\sigma) = (J^s_0\epsilon, J^r_x\sigma)\cdot(J^s_0\alpha, J^r_0 a)$, we simply take $(J^s_0\alpha, J^r_0 a) = (J^s_0(\epsilon^{-1}\circ\tilde\epsilon), J^r_0(\tau\circ(\sigma,\tilde\sigma)\circ\tilde\epsilon))$, where $\tau : P\times_M P\to G$ is given by the implicit equation $r(u_x, \tau(u_x, u'_x)) = u'_x$, where r is the principal right action on P and $u_x, u'_x \in p^{-1}(x)$. [1] Now Theorem 2.3 implies that the gauge natural prolongation $(W^{s,r}P, X, \tilde p, W^{s,r}_nG)$ of order (s,r) of the principal bundle P is a principal bundle too. We shortly denote $W^rP = W^{r,r}P$ and $W^r_nG = W^{r,r}_nG$ for all $r\in\mathbb{N}$.

We shall give another description of W^r_nG and W^rP. We shall deal with the category $\mathcal{PB}_n(G)$ consisting of principal bundles with n-dimensional bases and

[1] The mapping τ is well defined since the right action is free. We immediately get $\tau(u_x,u_x) = e$ and from

$$\begin{aligned}r(u_x\cdot a, \tau(u_x\cdot a, u'_x\cdot a')) &= u'_x\cdot a' = r(r(u_x,\tau(u_x,u'_x)), a')\\ &= r(u_x, \tau(u_x,u'_x)a') = r(u_x\cdot a, a^{-1}\tau(u_x,u'_x)a')\end{aligned}$$

we see that the mapping τ satisfies the equation $\tau(u_x\cdot a, u'_x\cdot a') = a^{-1}\tau(u_x,u'_x)a'$.

fixed structure group G, with $\mathcal{PB}_n(G)$-morphisms which cover local diffeomorphisms between the base manifolds. So a $\mathcal{PB}_n(G)$-morphism ψ from (P, p, X, G) into (P', p', X', G) is a smooth fibered map over a local diffeomorphism $\psi_0 : X \to X'$ satisfying $\psi \circ \rho_a = \rho'_a \circ \psi$ for all $a \in G$, where ρ and ρ' are the principal right actions on P and P'.

All $\mathcal{PB}_n(G)$-morphisms are local isomorphisms. Indeed, using some local trivializations ψ_α and $\psi'_{\alpha'}$ on P and P' respectively we have

$$\psi'_{\alpha'} \circ \psi \circ \psi_\alpha^{-1}(x, a) = ((\psi'_{\alpha'} \circ \psi \circ \psi_\alpha^{-1})_0(x), (\mathrm{pr}_2 \circ \psi'_{\alpha'} \circ \psi \circ \psi_\alpha^{-1}(x, e))a),$$

but $(\psi'_{\alpha'} \circ \psi \circ \psi_\alpha^{-1})_0$ must be a local diffeomorphism and the left translation through $\mathrm{pr}_2 \circ \psi'_{\alpha'} \circ \psi \circ \psi_\alpha^{-1}(x, e)$ is a diffeomorphism, thus a $\mathcal{PB}_n(G)$-morphism ψ is a local diffeomorphism. Locally for each $u' \in P'$ there exists one $u \in P$ such that $\psi(u) = u'$, hence we have

$$\begin{aligned}\psi^{-1}(u' \cdot a) = \psi^{-1}(\psi(u) \cdot a) = \psi^{-1}(\psi(u \cdot a)) = u \cdot a = \psi^{-1}(\psi(u)) \cdot a \\ = \psi^{-1}(u') \cdot a\end{aligned}$$

for each u' in some open set in P' and for all $a \in G$. Since ψ^{-1} is fiber respecting too, a $\mathcal{PB}_n(G)$-morphism ψ is really a local isomorphism.

We have a bijection between the set of $\mathcal{PB}_n(G)$-morphisms from $\mathbb{R}^n \times G$ into P and the set of pairs, which are formed by local diffeomorphisms and local sections of P. To a $\mathcal{PB}_n(G)$-morphism $\psi : \mathbb{R}^n \times G \to P$ we associate a local diffeomorphism ψ_0 and a local section ψ_1 of P by the relation $\psi(x, a) = (\psi_1 \circ \psi_0(x)) \cdot a$, i.e. $\psi_1(x') = \psi(\psi_0^{-1}(x'), e)$ and from $p \circ \psi_1(x') = p \circ \psi(\psi_0^{-1}(x'), e) = \psi_0 \circ \mathrm{pr}_1(\psi_0^{-1}(x'), e) = x'$ we see that ψ_1 is in fact a local section of P. Analogously, every $\mathcal{PB}_n(G)$-automorphism $\phi : \mathbb{R}^n \times G \to \mathbb{R}^n \times G$ is fully determined by its restriction $\phi_1 : \mathbb{R}^n \to G$, $\phi_1(x) = \mathrm{pr}_2 \circ \phi(x, e)$ and the underlying map $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$, i.e. we have $\phi(x, a) = (\phi_0(x), (\phi_1(x))a)$ (compare with Equation (3.15)). Now we can consider the group of r-jets at $(0, e)$ of all automorphisms $\phi : \mathbb{R}^n \times G \to \mathbb{R}^n \times G$ with $\phi_0(0) = 0$, where the multiplication μ is defined by the composition of jets, i.e. $\mu(J^r_{(0,e)}\phi, J^r_{(0,e)}\psi) = J^r_{(0,e)}(\phi \circ \psi)$. This definition is correct. In fact, we have

$$J^r_{(0,\phi_1(0))}\rho_a \circ J^r_{(0,e)}\phi = J^r_{(0,e)}(\rho_a \circ \phi) = J^r_{(0,e)}(\phi \circ \rho_a) = J^r_{(0,a)}\phi \circ J^r_{(0,e)}\rho_a \quad (2.5)$$

for all $a \in G$, where ρ now denotes the right action on $\mathbb{R}^n \times G$. But ρ_a is a diffeomorphism, thus we get $J^r_{(0,e)}\phi = J^r_{(0,e)}\tilde{\phi}$ iff $J^r_{(0,a)}\phi = J^r_{(0,a)}\tilde{\phi}$. Hence for $J^r_{(0,e)}\phi = J^r_{(0,e)}\tilde{\phi}$ and $J^r_{(0,e)}\psi = J^r_{(0,e)}\tilde{\psi}$ the computation

$$\begin{aligned}\mu(J^r_{(0,e)}\tilde{\phi}, J^r_{(0,e)}\tilde{\psi}) = J^r_{(0,e)}(\tilde{\phi} \circ \tilde{\psi}) = J^r_{(0,\tilde{\psi}_1(0))}\tilde{\phi} \circ J^r_{(0,e)}\tilde{\psi} = J^r_{(0,\psi_1(0))}\phi \circ J^r_{(0,e)}\psi \\ = \mu(J^r_{(0,e)}\phi, J^r_{(0,e)}\psi)\end{aligned}$$

shows that the map μ is well defined, further the composition of jets is smooth, the unit is $J^r_{(0,e)}\mathrm{id}_{\mathbb{R}^n \times G}$ and the inverse elements are the jets of inverse maps, which always exist locally. Therefore the multiplication μ is indeed correctly defined. For every $\phi, \psi \in \mathrm{Hom}_{\mathcal{PB}_n(G)}(\mathbb{R}^n \times G, \mathbb{R}^n \times G)$ we have

$$\phi \circ \psi(x,a) = \phi(\psi_0(x), (\psi_1(x))a) = (\phi_0 \circ \psi_0(x), \phi_1(\psi_0(x))(\psi_1(x))a).$$

If we write $(A,B) = (J_0^r\phi_0, J_0^r\phi_1) \in W_n^rG$ for the element corresponding to jet $J^r_{(0,e)}\phi$ and $(C,D) = (J_0^r\psi_0, J_0^r\psi_1) \in W_n^rG$ for the element corresponding to jet $J^r_{(0,e)}\psi$, then in this identification we get

$$\mu((A,B),(C,D)) = (A \circ C, (B \circ C)D),$$

where on the right hand side in the second input we use the multiplication in T_n^rG. If we compare this multiplication μ with the one on W_n^rG defined previously, then we see that there is an isomorphism between the group considered now and the principal prolongation W_n^rG of Lie group G.

We can define the set $\{J^r_{(0,e)}\psi \in J^r(\mathbb{R}^n \times G, P) : \psi \in \mathrm{Hom}_{\mathcal{PB}_n(G)}(\mathbb{R}^n \times G, P)\}$ and we denote it by the same symbol as the gauge natural prolongation W^rP. There is a bijection between $\mathbb{R}^n \times W_n^rG$ and $W^r(\mathbb{R}^n \times G)$

$$\mathbb{R}^n \times W_n^rG \ni (x, J^r_{(0,e)}\phi) \mapsto J^r_{(0,e)}(\tau_x \circ \phi) \in W^r(\mathbb{R}^n \times G),$$

where we use the translation t_x to define $\tau_x = t_x \times \mathrm{id}_G$, with the inverse map given by $J^r_{(0,e)}\tilde{\phi} \mapsto (\mathrm{pr}_1 \circ \tilde{\phi}(0,e), J^r_{(0,e)}(\tau^{-1}_{\mathrm{pr}_1 \circ \tilde{\phi}(0,e)} \circ \tilde{\phi}))$. Thus there is a structure of smooth manifold on $W^r(\mathbb{R}^n \times G)$. If we define the functor W^r on $\mathcal{PB}_n(G)$-morphisms as the composition of jets by $W^r\chi(J^r_{(0,e)}\psi) = J^r_{(0,e)}(\chi \circ \psi)$, then we can transform any principal bundle atlas on P to an atlas on W^rP. The right action of W_n^rG on W^rP is given again by jet composition, i.e. $J^r_{(0,e)}\psi \cdot J^r_{(0,e)}\phi = J^r_{(0,e)}(\psi \circ \phi)$, which is free and transitive on fibers (all jets are invertible), therefore $(W^rP, p \circ \beta, X, W_n^rG)$ is a principal bundle. If we write $(u,v) = (J_0^r\psi_0, J^r_{\psi_0(0)}\psi_1) \in W^rP$ for the element corresponding to jet $J^r_{(0,e)}\psi$ and $(A,B) = (J_0^r\phi_0, J_0^r\phi_1) \in W_n^rG$ for the element corresponding to jet $J^r_{(0,e)}\phi$, then from the computation

$$\begin{aligned}\psi \circ \phi(x,a) &= \psi(\phi_0(x), \phi_1(x)a) = (\psi_1 \circ \psi_0 \circ \phi_0(x)) \cdot (\phi_1(x)a)\\ &= (\rho \circ (\psi_1, \phi_1 \circ \phi_0^{-1} \circ \psi_0^{-1}) \circ (\psi_0 \circ \phi_0)(x)) \cdot a\end{aligned}$$

(in such an identification) we get $(u,v) \cdot (A,B) = (u \circ A, v \cdot (B \circ A^{-1} \circ u^{-1}))$, where $\cdot$ on the right hand side is as in (2.3). This corresponds to the right action of W_n^rG on W^rP defined previously [2].

Finally we prove that $P \mapsto W^{s,r}P$ for $P \in \mathrm{Ob}(\mathcal{PB}_n(G))$, $\psi \mapsto W^{s,r}\psi = (F^s\psi_0, J^r\psi)$ for $\psi \in \mathrm{Hom}_{\mathcal{PB}_n(G)}(P,P')$ is a functor $\mathcal{PB}_n(G) \to \mathcal{PB}_n(W_n^{s,r}G)$.

[2] A similar construction using the concept (r,s,q)-jet can be done also for $W^{s,r}P$. We say that two maps $f, g \in \mathrm{Hom}_{\mathcal{FM}}(Y,Y')$ determine the same (r,s,q)**-jet** at $y \in Y$, $s, q \geq r$, if

$$J_y^rf = J_y^rg \text{ and } J_y^sf|_{Y_x} = J_y^sg|_{Y_x} \text{ and } J_x^qBf = J_x^qBg,$$

where Y_x is the fiber, which contains y. Then $W^{s,r}P$ can be identified with the space of all (r,r,s)-jets at $(0,e)$ of a homomorphism $f \in \mathrm{Hom}_{\mathcal{PB}_n(G)}(\mathbb{R}^n \times G, P)$.

The functoriality follows from $W^{s,r} = \times_{B(\ )} \circ (F^s \circ B, J^r)$. It remains to prove that $W^{s,r}\psi \in \mathrm{Hom}_{\mathcal{PB}_n(W_n^{s,r}G)}(W^{s,r}P, W^{s,r}P')$. The computation for $u = J_0^s \epsilon \in F^s X$, $v = J_x^r \sigma \in J^r P$, $A = J_0^s a \in L_n^s$, $B = J_0^r b \in T_n^r G$

$$\begin{aligned}
W^{s,r}\psi((u,v)\cdot(A,B)) &= W^{s,r}\psi(u\circ A, v\cdot(B\circ\pi^{s,r}(A^{-1}\circ u^{-1})))\\
&= (F^s\psi_0(u\circ A), J^r\psi(v\cdot(B\circ\pi^{s,r}(A^{-1}\circ u^{-1}))))\\
&= (J_0^s(\psi_0\circ\epsilon\circ a), J^r_{\psi_0(x)}(\psi\circ(\sigma\cdot(b\circ a^{-1}\circ\epsilon^{-1}))\circ\psi_0^{-1}))\\
&= (J_0^s(\psi_0\circ\epsilon\circ a), J^r_{\psi_0(x)}((\psi\circ\sigma\circ\psi_0^{-1})\cdot(b\circ a^{-1}\circ\epsilon^{-1}\circ\psi_0^{-1})))\\
&= (F^s\psi_0(u)\circ A, J^r\psi(v)\cdot(B\circ\pi^{s,r}(A^{-1}\circ(F^s\psi_0(u))^{-1})))\\
&= (F^s\psi_0(u), J^r\psi(v))\cdot(A,B) = W^{s,r}\psi(u,v)\cdot(A,B)
\end{aligned}$$

shows that $W^{s,r}\psi$ is indeed a $\mathcal{PB}_n(W_n^{s,r}G)$- morphism.

For every bundle $P\times_l Z$ associated to a principal bundle (P,p,X,G) there is a canonical left action $l^r : W_n^r G \times T_n^r Z \to T_n^r Z$ given by

$$l^r(J^r_{(0,e)}\phi, J_0^r s) = J_0^r(l\circ(\phi_1\circ\phi_0^{-1}, s\circ\phi_0^{-1})), \tag{2.6}$$

i.e. as the composition of the prolonged action $T_n^r l : T_n^r G \times T_n^r Z \to T_n^r Z$, which is defined analogously as the multiplication in Example 2.2, and the canonical left action of L_n^r on both $T_n^r G$ and $T_n^r Z$. We denote by $\times_l Z$ the functor, which associates to any principal fiber bundle homomorphism $\Phi : P \to P'$ a homomorphism $[\Phi, \mathrm{id}_Z] = \Phi\times_l Z : P\times_l Z \to P'\times_l Z$, $[y,z]\mapsto[\Phi(y),z]$.

Theorem 2.4. *There is an isomorphism $J^r(P\times_l Z)\cong W^r P\times_{l^r} T_n^r Z$. In this identification the correspondence $P\mapsto J^r(P\times_l Z)$, $\Phi\mapsto J^r([\Phi,\mathrm{id}_Z]) = [F^r\circ B(\Phi)\times J^r(\Phi), \mathrm{id}_{T_n^r Z}]$ is a functor from $\mathcal{PB}_n(G)$ to the subcategory of $\mathcal{FM}_n$, which is formed by bundles associated to an r-th principal prolongation of a principal G-bundle and their homomorphisms.*

Proof: See [25], [34], [32].

$\diamondsuit$

The functor from Theorem 2.4 is one example of the following concept. A **gauge natural bundle functor** or **G-natural bundle functor** over n-dimensional manifolds is a functor $F : \mathcal{PB}_n(G) \to \mathcal{FM}$ such that:

1. every principal bundle $p : P \to BP$ from $\mathrm{Ob}(\mathcal{PB}_n(G))$ is transformed into a fibered manifold $q_P : FP \to BP$,

2. every principal fiber bundle homomorphism $f\in \mathrm{Hom}_{\mathcal{PB}_n(G)}(P,P')$ is transformed into a morphism of fibered manifolds $Ff : FP\to FP'$ over Bf,

3. for every open subset $U\in BP$, the inclusion $i : p^{-1}(U)\to P$ is transformed into the inclusion $Fi : q_P^{-1}(U)\to FP$.

If F is a gauge natural bundle functor and $(P, p, X, G) \in \mathrm{Ob}(\mathcal{PB}_n(G))$, then $q_P : FP \to X$ will be called a **gauge natural bundle**.

Example 2.6. A **natural bundle functor** over n-dimensional manifolds is a functor $\bar{F} : \mathcal{M}f_n \to \mathcal{FM}$ such that:

1. $B \circ \bar{F} = \mathrm{Id}_{\mathcal{M}f_n}$, i.e. the projections form a natural transformation $p : \bar{F} \to \mathrm{Id}_{\mathcal{M}f_n}$,
2. if $i : U \to X$ is an inclusion of an open submanifold, then $\bar{F}U = p_X^{-1}(U)$ and $\bar{F}i : p_X^{-1}(U) \to \bar{F}X$ is the inclusion.

From a natural bundle functor $\bar{F}$ we simply obtain a gauge natural bundle functor F by $F = \bar{F} \circ B$. Conversely, the choice $G = \{e\}$ makes from a gauge natural bundle functor a natural bundle functor.

Example 2.7. The functor $(\times_l S) \circ W^r : \mathcal{PB}_n(G) \to \mathcal{FM}$ is a gauge natural bundle functor.

An argument analogous to the one in Example 2.5 using (2.5) shows that $J^r_y f = J^r_y g$ for $f, g \in \mathrm{Hom}_{\mathcal{PB}_n(G)}(P, P')$ and $y \in P_x$, $x \in BP$ implies $J^r_z f = J^r_z g$ for all $z \in P_x$ in the fiber P_x over x. In this case we write $\mathbf{J}^r_x f = \mathbf{J}^r_x g$ and say that f and g have the same **fiber r-jet at** x. A gauge natural bundle functor is said to be of **order** r, if $\mathbf{J}^r_x f = \mathbf{J}^r_x g$ implies $Ff|_{F_x P} = Fg|_{F_x P}$, where we denote by $F_x P = (FP)_x$ the fiber over x. A gauge natural bundle functor is said to be **regular**, if every smoothly parametrized family of $\mathcal{PB}_n(G)$-morphisms

$$f : X \to \mathrm{Hom}_{\mathcal{PB}_n(G)}(P, P')$$

parametrized by a manifold X (i.e. the map $X \times P \to P'$, $(t, u) \mapsto f(t)(u)$ is smooth) is transformed into a smoothly parametrized family of fibered manifold morphisms

$$\tilde{F}f : X \to \mathrm{Hom}_{\mathcal{FM}}(FP, FP')$$

(i.e. the map $X \times FP \to FP'$, $(t, v) \mapsto F(f(t))(v)$ is smooth).

Theorem 2.5. *For every r-th order regular gauge natural bundle functor $F : \mathcal{PB}_n(G) \to \mathcal{FM}$ there is a canonical structure of an associated bundle $W^r P \times_l S$ on FP given by a map b_P and the values of the functor F in this identification lie in the category of associated bundles and their homomorphisms, i.e. we have the natural equivalence $b : [W^r, \mathrm{id}_S] \to F$ and the following diagram commutes:*

$$\begin{array}{ccc} W^r P \times_l S & \xrightarrow{b_P} & FP \\ \Big\downarrow{\scriptstyle [W^r f, \mathrm{id}_S]} & & \Big\downarrow{\scriptstyle Ff} \\ W^r P' \times_l S & \xrightarrow{b_{P'}} & FP' \end{array}$$

Proof: First we define the **associated maps** $F_{P,P'} : \operatorname{inv} J^r(P,P') \times_{BP} FP \to FP'$ of the gauge natural bundle functor F by $F_{P,P'}(J^r_z f, y) = Ff(y)$. This is well defined. Indeed, $J^r_z f = J^r_z g$ with $z \in P_x$ implies $\mathbf{J}^r_x f = \mathbf{J}^r_x g$. Since F is of order r, we have

$$F_{P,P'}(J^r_z g, y) = Fg(y) = Fg|_{F_xP}(y) = Ff|_{F_xP}(y) = Ff(y) = F_{P,P'}(J^r_z f, y)$$

and the associated maps of the gauge natural bundle functor F are really well defined. To prove that these associated maps are smooth, it is sufficient to restrict ourselves to $P = P' = \mathbb{R}^{n+m}$. We consider the map $\mathrm{ev} : \operatorname{inv} J^r(\mathbb{R}^{n+m}, \mathbb{R}^{n+m}) \times \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ given by $\mathrm{ev}(J^r_z f, u) = \tilde{f}(u)$, where $\tilde{f}$ is the canonical polynomial (as in the proof of Theorem 2.2), which corresponds to $J^r_z f$. We see that the map ev is well defined and smooth and by regularity of F we get that $\tilde{F}\mathrm{ev}$ is smooth too. The computation

$$\begin{aligned}(\tilde{F}\mathrm{ev})|_{\operatorname{inv} J^r(\mathbb{R}^{n+m},\mathbb{R}^{n+m})\times_{\mathbb{R}^n}\mathbb{R}^{n+m}}(J^r_z f, y) &= F(\mathrm{ev}_{J^r_z f})(y) = F(\tilde{f})(y)\\ &= F_{\mathbb{R}^{n+m},\mathbb{R}^{n+m}}(J^r_z \tilde{f}, y) = F_{\mathbb{R}^{n+m},\mathbb{R}^{n+m}}(J^r_z f, y)\end{aligned}$$

shows that $F_{\mathbb{R}^{n+m},\mathbb{R}^{n+m}}$ is smooth, thus the associated maps are smooth.

Now we can define the smooth **induced action** of the r-th principal prolongation $W^r_n G$ of a Lie group G on the standard fiber $S = F_0(\mathbb{R}^n \times G)$ by $l = F_{\mathbb{R}^n\times G,\mathbb{R}^n\times G}|_{W^r_n G\times S}$. The map $b_P : W^r P \times_l S \to FP$ given by the factorization of $F_{\mathbb{R}^n\times G,P}|_{W^rP\times S}$ through the surjective submersion $q : W^r P \times S \to W^r P \times_l S$ is well defined. In fact, we have

$$\begin{aligned}b_P(J^r_{(0,e)}\psi \cdot J^r_{(0,e)}\phi, l(J^r_{(0,e)}\phi^{-1}, s)) &= b_P(J^r_{(0,e)}(\psi\circ\phi), F\phi^{-1}(s))\\ &= F(\psi\circ\phi)\circ F\phi^{-1}(s) = F\psi(s) = b_P(J^r_{(0,e)}\psi, s).\end{aligned}$$

for all $J^r_{(0,e)}\psi \in W^r P$, $J^r_{(0,e)}\phi \in W^r_n G$ and $s \in S$. From $b_P \circ q = F_{\mathbb{R}^n\times G,P}|_{W^rP\times S}$ we obtain by universal property of surjective submersion that b_P is smooth. The map $F_{P,\mathbb{R}^n\times G}|_{W^rP\times F_{\psi_0(0)}P}(u^{-1},\ ) : F_{\psi_0(0)}P \to S$ for $u = J^r_{(0,e)}\psi$ is the inverse map to $F_{\mathbb{R}^n\times G,P}|_{W^rP\times S}(u,\ ) : S \to F_{\psi_0(0)}P$ and we denote this diffeomorphism from a fiber of FP to the standard fiber S by c_u. Then the map $b_P^{-1} : FP \to W^r P \times_l S$, $b_P^{-1}(y) = [u, c_u(y)]$ with $u \in W^r_{q_P(y)}P$ is the inverse map to b_P. Locally for some section s_α of $p_P : W^r P \to BP$ we can write $b_P^{-1}(y) = [s_\alpha \circ q_P(y), c_{s_\alpha\circ q_P(y)}(y)]$, because from $p_P \circ s_\alpha \circ q_P(y) = q_P(y)$ we see that $s_\alpha \circ q_P(y) \in W^r_{q_P(y)}P$, therefore b_P^{-1} is smooth and b_P is a diffeomorphism. This b_P is the isomorphism $W^r P \times_l S \cong FP$.

Finally the calculation

$$\begin{aligned}Ff\circ b_P([J^r_{(0,e)}\psi, s]) &= Ff\circ F\psi(s) = F(f\circ\psi)(s) = b_{P'}([J^r_{(0,e)}(f\circ\psi), s])\\ &= b_{P'}([W^r f(J^r_{(0,e)}\psi), s]) = b_{P'}\circ[W^r f, \mathrm{id}_S]([J^r_{(0,e)}\psi, s])\end{aligned}$$

finishes the proof.

$\diamondsuit$

It can be proved that every gauge natural bundle functor is regular (see [25]), thus Theorem 2.5 says that every r-th order gauge natural bundle is of the form as in Example 2.7. Quite similarly, using the concept of (r, s, q)-jets, it can be proved that every gauge natural bundle functor of order (s, r) [3] is a fiber bundle associated to $W^{s,r}$. We will call a fiber bundle associated to $W^{s,r}P$ the gauge natural bundle too.

Example 2.8. It is possible to introduce connections in several equivalent ways. We will describe connections as sections of the first jet prolongation. We show that the bundle of principal connections is a gauge natural bundle.

Consider the principal action $r : P \times G \to P$ on (P, p, X, G). Then we have the canonical right action $\tilde{r} : J^1P \times G \to J^1P, \tilde{r}(J^1_x s, g) = J^1_x(r_g \circ s)$. Then a principal connection Γ on P can be considered as a G-equivariant section $\Gamma : P \to J^1P$ of the first jet prolongation $\beta : J^1P \to P$. In fact, from [4]

$$r_g^*\Phi(\xi_u) = (T_u r_g)^{-1}(T_u r_g \cdot \xi_u - \Gamma(u \cdot g) \circ T_u(p \circ r_g) \cdot \xi_u),$$
$$\Phi(\xi_u) = (T_u r_g)^{-1}(T_u r_g \cdot \xi_u - \tilde{r}_g \circ \Gamma(u) \circ T_u p \cdot \xi_u)$$

for $g \in G$ and $u \in P$ we see that the connection Φ considered as a vector valued one form is principal ($r_g^*\Phi = \Phi$ holds for all $g \in G$) iff Γ is a G-equivariant section of the first jet prolongation $\beta : J^1P \to P$. The **bundle of principal connections** is defined as $QP = J^1P/G$. Now we show that the sections of QP are in bijection with the principal connections on P. The following two theorems can be found in [25].

Theorem 2.6. *The functor $Q : \mathcal{PB}_n \to \mathcal{FM}_n$ associates with each principal fiber bundle (P, p, X, G) the fiber bundle QP over the base X with the standard fiber $J^1_0(\mathbb{R}^n, G)_e$. The smooth sections of QP are in bijection with the principal connections on P.*

Proof: From the definition of $\tilde{r}$ we see that the source projection $\alpha : J^1P \to X$ factors trough $q : J^1P \to J^1P/G$ to $\tilde{\alpha} : J^1P/G \to X$ so that $\alpha = \tilde{\alpha} \circ q$. For a homomorphism of principal bundles $(\phi, \varphi) : (P, p, X, G) \to (\bar{P}, \bar{p}, \bar{X}, \bar{G})$ over a homomorphism $\varphi : G \to \bar{G}$ the relation

$$J^1\phi(\tilde{r}_g(J^1_x s)) = J^1_{\phi_0(x)}(\bar{r}_{\varphi(g)} \circ \phi \circ s \circ \phi_0^{-1})$$

holds for all $J^1_x s \in J^1P$ and $g \in G$. Hence the map $Q\phi : QP \to Q\bar{P}, [J^1_x s] \mapsto [J^1\phi(J^1_x s)]$ is well defined. If we define a suitable smooth structure on QP

[3] As before $J^{r,r,s}_y f = J^{r,r,s}_y g$ at a point $y \in P_x$ for $f, g \in \mathrm{Hom}_{\mathcal{PB}_n(G)}(P, P')$ implies that $J^{r,r,s}_y f = J^{r,r,s}_y g$ holds for all $y \in P_x$ and we write $\mathbf{J}^{r,r,s}_x f = \mathbf{J}^{r,r,s}_x g$. We say that a gauge natural bundle functor F is of **order** (s, r), $s \geq r$, if $\mathbf{J}^{r,r,s}_x f = \mathbf{J}^{r,r,s}_x g$ implies $Ff|_{F_x P} = Fg|_{F_x P}$.

[4] For a vector bundle (Y, π, X) we denote the space of Y-valued k-forms by $\Omega^k(X; Y) = \Gamma(\wedge^k T^*X \otimes Y)$. If $f : X \to \bar{X}$ is a local diffeomorphism, we can consider the pullback $f^* : \Omega^k(\bar{X}; T\bar{X}) \to \Omega^k(X; TX)$, given by

$$(f^*\lambda)_x(\xi_1, \ldots, \xi_k) = (T_x f)^{-1}\lambda_{f(x)}(T_x f \cdot \xi_1, \ldots, T_x f \cdot \xi_k).$$

than we have proved the first part of the theorem. Let us first assume $P = \mathbb{R}^n \times G$. We have the identification $J^1(\mathbb{R}^n \times G) \cong \mathbb{R}^n \times J^1_0(\mathbb{R}^n, G), J^1_x s \mapsto (x, J^1_0(\mathrm{pr}_2 \circ s \circ t_x))$. If we use a canonical representative in each orbit $[J^1_x s]$ with $s(x) = (x, e)$, $e \in G$ being the unit, then we get the induced smooth structure $Q(\mathbb{R}^n \times G) \cong \mathbb{R}^n \times J^1_0(\mathbb{R}^n, G)_e \cong \mathbb{R}^n \times \mathbb{R}^{nm}$, $\dim G = m$. Now we see that q becomes a surjective submersion. From the universal property of surjective submersions we see that $\tilde{\alpha}$ is smooth. From the fact that the composition of jets is smooth and using universal property of q again we see that $Q\phi$ is smooth for every $\mathcal{PB}_n$-morphism $\phi : \mathbb{R}^n \times G \to \mathbb{R}^n \times \bar{G}$. For every principal fiber bundle atlas (U_α, ϕ_α) on (P, p, X, G) the charts $(U_\alpha, Q\phi_\alpha)$ form a fiber bundle atlas on $(QP, \tilde{\alpha}, X, J^1_0(\mathbb{R}^n, G)_e)$. The functoriality of Q follows from the functoriality of J^1.

Now we prove the second part of the theorem. To each principal connection Γ on P we associate a section $S = q \circ \Gamma \circ s$ of QP, where s is an arbitrary section of P. As a composition of smooth mappings S is smooth. Conversely, to each section S of QP we associate a principal connection Γ on P defined by $\Gamma(u) = \tilde{r}_g \circ i \circ S \circ p(u)$ for each $u \in P$, where i is a map such that $q \circ i = \mathrm{id}_{QP}$ and $g \in G$ is given by the condition $u = r_g \circ \beta \circ i \circ S \circ p(u)$. Since we can write $\Gamma = \tilde{r}_{\tau(\beta \circ i \circ S \circ p(), \mathrm{id}_P())} \circ i \circ S \circ p$, where τ was defined on p. 11, and the surjective submersion q admits smooth sections, i.e. we can take a smooth i at least locally, Γ is smooth too. These associations are inverse to each other.

$\diamondsuit$

Now we show that the bundle of principal connections is a gauge natural bundle of order 1. This is the special case of Theorem 2.5.

Theorem 2.7. *$QP \cong W^1P \times_{l_2} S$, where $S = (Q(\mathbb{R}^n \times G))_0$ and the action l_2 is given by $l_2 : W^1_n G \times S \to S$, $l_2(J^1_{(0,e)}\phi, Y) = Q\phi(Y)$.*

Proof: From $J^1_{(0,e)}\phi = J^1_{(0,e)}\tilde{\phi}$ we see that $J^1_0\phi_0^{-1} = J^1_0\tilde{\phi}_0^{-1}$ so $Q\phi(Y) = [J^1\phi(J^1_0 s)] = [J^1_0(\phi \circ s \circ \phi_0^{-1})] = [J^1_{(0,e)}\phi \circ J^1_0 s \circ J^1_0 \phi_0^{-1}] = [J^1_{(0,e)}\tilde{\phi} \circ J^1_0 s \circ J^1_0 \tilde{\phi}_0^{-1}] = Q\tilde{\phi}(Y)$, where we take $Y = J^1_0 s$ with $s(0) = (0, e)$, so l_2 is well defined. Because the composition of jets is smooth we see that l_2 is smooth too. We see that $q : W^1P \times S \to QP$, $q(J^1_{(0,e)}\psi, Y) = Q\psi(Y)$ is well defined, moreover we can factorize q to smooth $\tilde{q} : W^1P \times_{l_2} S \to QP$. In fact, this is well defined, because we have $\tilde{q}([J^1_{(0,e)}\psi \cdot J^1_{(0,e)}\phi, J^1_{(0,e)}\phi^{-1} \cdot Y]_{l_2}) = q(J^1_{(0,e)}(\psi \circ \phi), Q\phi^{-1}(Y)) = Q\psi \circ Q\phi \circ Q\phi^{-1}(Y) = Q\psi(Y) = \tilde{q}([J^1_{(0,e)}\psi, Y]_{l_2})$ and it is smooth because of the universal property of surjective submersions. Further $\tilde{q}$ is a bijection with the inverse $\tilde{q}^{-1}(Y) = [J^1_{(0,e)}\psi, Q\psi^{-1}(Y)]_{l_2}$ for Y in a fiber chart $(U, Q\psi)$. From $\tilde{q}_0 = \mathrm{id}_X$ and because $\tilde{q}$ looks in canonical local trivializations like the identity we see that $\tilde{q}$ is an isomorphism.

$\diamondsuit$

If we have more interactions and more matter particle species each corresponding to some gauge natural bundle, then it is good to know that their fiber product is a gauge natural bundle too, and what it looks like.

Theorem 2.8. $(W^{s_1,r_1}P \times_{l_1} S_1) \times_X (W^{s_2,r_2}P \times_{l_2} S_2) \cong W^{s,r}P \times_l (S_1 \times S_2)$ *where* $r = \max\{r_1, r_2\}$, $s = \max\{s_1, s_2\}$ *and* l *is induced by* l_1 *and* l_2.

Proof: We define a map

$$
\begin{aligned}
b : (W^{s_1,r_1}P \times_{l_1} S_1) \times_X (W^{s_2,r_2}P \times_{l_2} S_2) \to W^{s,r}P \times_l (S_1 \times S_2),\\
([(J_0^{s_1}\epsilon_1, J_x^{r_1}\sigma_1), f_1]_{l_1}, [(J_0^{s_2}\epsilon_2, J_x^{r_2}\sigma_2), f_2]_{l_2}) \mapsto\\
[(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}), (g_1^{-1} \cdot_{l_1} f_1, g_2^{-1} \cdot_{l_2} f_2)]_l
\end{aligned}
$$

where $\iota(s) = i$ for $s = s_i$ $(i = 1, 2)$ and similarly for r, the g_i's are given by the condition: there exists a unique $g_i \in W_n^{s_i,r_i}G$ such that

$$\pi^{s,s_i} \times \pi^{r,r_i}(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}) = (J_0^{s_i}\epsilon_i, J_x^{r_i}\sigma_i) \cdot g_i$$

and the left action l is defined by formula

$$g \cdot_l (f_1, f_2) = (\pi^1(g) \cdot_{l_1} f_1, \pi^2(g) \cdot_{l_2} f_2),$$

where π^i are the homomorphisms $\pi^i : W_n^{s,r}G \to W_n^{s_i,r_i}G$ given by the canonical projections, and since l_i are left actions, l is a left action too.

Now we prove that b is well defined. For another pair of representatives we get

$$
\begin{aligned}
&([(J_0^{s_1}\epsilon_1, J_x^{r_1}\sigma_1) \cdot h_1, h_1^{-1} \cdot_{l_1} f_1]_{l_1}, [(J_0^{s_2}\epsilon_2, J_x^{r_2}\sigma_2) \cdot h_2, h_2^{-1} \cdot_{l_2} f_2]_{l_2}) \mapsto\\
&\qquad [(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}) \cdot h, (\bar{g}_1^{-1} \cdot_{l_1} h_1^{-1} \cdot_{l_1} f_1, \bar{g}_2^{-1} \cdot_{l_2} h_2^{-1} \cdot_{l_2} f_2)]_l
\end{aligned}
$$

for some $h \in W_n^{s,r}G$ and from

$$
\begin{aligned}
(J_0^{s_i}\epsilon_i, J_x^{r_i}\sigma_i) \cdot g_i \cdot \pi^i h &= \pi^{s,s_i} \times \pi^{r,r_i}(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}) \cdot \pi^i h\\
&= \pi^{s,s_i} \times \pi^{r,r_i}((J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}) \cdot h) = (J_0^{s_i}\epsilon_i, J_x^{r_i}\sigma_i) \cdot h_i \cdot \bar{g}_i
\end{aligned}
$$

we get $\bar{g}_i = h_i^{-1} \cdot g_i \cdot \pi^i h$ so

$$
\begin{aligned}
&[(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}) \cdot h, (\bar{g}_1^{-1} \cdot_{l_1} h_1^{-1} \cdot_{l_1} f_1, \bar{g}_2^{-1} \cdot_{l_2} h_2^{-1} \cdot_{l_2} f_2)]_l\\
&= [(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}) \cdot h, (\pi^1 h^{-1} \cdot_{l_1} g_1^{-1} \cdot_{l_1} f_1, \pi^2 h^{-1} \cdot_{l_2} g_2^{-1} \cdot_{l_2} f_2)]_l\\
&= [(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}) \cdot h, h^{-1} \cdot_l (g_1^{-1} \cdot_{l_1} f_1, g_2^{-1} \cdot_{l_2} f_2)]_l\\
&= [(J_0^{s}\epsilon_{\iota(s)}, J_x^{r}\sigma_{\iota(r)}), (g_1^{-1} \cdot_{l_1} f_1, g_2^{-1} \cdot_{l_2} f_2)]_l.
\end{aligned}
$$

So b is in fact well defined.

Now we prove that b is a bijection. We define a map

$$
\begin{aligned}
b^{-1} : W^{s,r}P \times_l (S_1 \times S_2) \to (W^{s_1,r_1}P \times_{l_1} S_1) \times_X (W^{s_2,r_2}P \times_{l_2} S_2),\\
[(J_0^{s}\epsilon, J_x^{r}\sigma), (f_1, f_2)]_l \mapsto\\
([\pi^{s,s_1} \times \pi^{r,r_1}(J_0^{s}\epsilon, J_x^{r}\sigma), f_1]_{l_1}, [\pi^{s,s_2} \times \pi^{r,r_2}(J_0^{s}\epsilon, J_x^{r}\sigma), f_2]_{l_2}).
\end{aligned}
$$

b^{-1} is well defined because

$$\begin{aligned}
&[(J_0^s\epsilon, J_x^r\sigma)\cdot h, h^{-1}\cdot_l (f_1,f_2)]_l \mapsto \\
&\quad([\pi^{s,s_1}\times\pi^{r,r_1}(J_0^s\epsilon, J_x^r\sigma)\cdot\pi^1 h, (\pi^1 h)^{-1}\cdot_{l_1} f_1]_{l_1}, \\
&\qquad [\pi^{s,s_2}\times\pi^{r,r_2}(J_0^s\epsilon, J_x^r\sigma)\cdot\pi^2 h, (\pi^2 h)^{-1}\cdot_{l_2} f_2]_{l_2}) \\
&= ([\pi^{s,s_1}\times\pi^{r,r_1}(J_0^s\epsilon, J_x^r\sigma), f_1]_{l_1}, [\pi^{s,s_2}\times\pi^{r,r_2}(J_0^s\epsilon, J_x^r\sigma), f_2]_{l_2}).
\end{aligned}$$

Immediately we see that $b^{-1}\circ b = \mathrm{id}$ and $b\circ b^{-1} = \mathrm{id}$.

Finally we must prove that b is a bundle isomorphism. So b must be a diffeomorphism and its projection on the base b_0 must be a diffeomorphism and the following diagram must be commutative:

$$\begin{array}{ccc}
(W^{s_1,r_1}P\times_{l_1} S_1)\times_X (W^{s_2,r_2}P\times_{l_2} S_2) & \xrightarrow{b} & W^{s,r}P\times_l (S_1\times S_2) \\
\big\downarrow{\scriptstyle \bar p_{12}} & & \big\downarrow{\scriptstyle \bar p} \\
X & \xrightarrow{b_0} & X
\end{array}$$

From the definition of b we see that the diagram is commutative and $b_0 = \mathrm{id}_X$, so b_0 is a diffeomorphism. We recall local trivializations, which we need. Let $\phi_\alpha^{-1} : U_\alpha\times W_n^{s,r}G\to p^{-1}(U_\alpha)$ be given by $\phi_\alpha^{-1}(x,a) = s_\alpha(x)\cdot a$ (similarly for $W^{s_i,r_i}P$ with the corresponding indices i). Let $\psi_\alpha^{-1} : U_\alpha\times S\to\bar p^{-1}(U_\alpha)$ be given by $\psi_\alpha^{-1}(x,f) = [\phi_\alpha^{-1}(x,e),f]_l$, where we denote $S = S_1\times S_2$ (similarly for $W^{s_i,r_i}P\times_{l_i} S_i$ with the corresponding indices i). Let for the fiber product $\psi_{12\alpha}^{-1} : U_\alpha\times S\to\bar p_{12}^{-1}(U_\alpha)$ be given by $\psi_{12\alpha}^{-1}(x,f_1,f_2) = (\psi_{1\alpha}^{-1}(x,f_1),\psi_{2\alpha}^{-1}(x,f_2))$. If we consider the sections related by $s_{i\alpha} = \pi^{s,s_i}\times\pi^{r,r_i}\circ s_\alpha$, then the computation

$$\begin{aligned}
\psi_\alpha\circ b\circ\psi_{12\alpha}^{-1}(x,f_1,f_2) &= \psi_\alpha\circ b([s_{1\alpha}(x),f_1]_{l_1},[s_{2\alpha}(x),f_2]_{l_2}) \\
&= \psi_\alpha([s_\alpha(x),f_1,f_2]_l) = (x,f_1,f_2)
\end{aligned}$$

shows that b locally looks like $\mathrm{id}_{U_\alpha\times S}$, so b is a local diffeomorphism, thus b is a diffeomorphism, because we have proved that b is a bijection.

$\diamondsuit$

2.2 Gauge Natural Operators

Let F and E be two gauge natural bundle functors over n-dimensional manifolds. A **gauge natural operator** $D : F\to E$ is a system of regular operators $D_P : \Gamma FP\to\Gamma EP$ for all $PB_n(G)$-objects $p : P\to BP$ such that

1. $D_{\bar P}(Ff\circ s\circ Bf^{-1}) = Ef\circ D_P s\circ Bf^{-1}$ for every section $s\in\Gamma FP$ and every $\mathcal{PB}_n(G)$-isomorphism $f : P\to\bar P$,

2. $D_{p^{-1}(U)}(s|U) = (D_P s)|U$ for every section $s\in\Gamma FP$ and every open subset $U\subset BP$

Here regular means that every smoothly parametrized family of sections is transformed into a smoothly parametrized family of sections. If, moreover, for a certain $k \in \mathbb{N} \cup \infty$ we find that for every $x \in M$, for every P and $s, q \in \Gamma FP$ the implication $J^k_x s = J^k_x q \;\Rightarrow D_P s(x) = D_P q(x)$ holds, then we say that D is of **order** k.

We present the following two theorems which can be found in [25], the proofs are analogous to the proofs for natural bundles in [34].

Theorem 2.9. *The k-th order gauge natural operators $F \to E$ are in canonical bijection with the natural transformations $J^k F \to E$.*

Proof: For every k-th order gauge natural operator $D : F \to E$ we define the natural transformation $\mathcal{D} : J^k F \to E$ by $\mathcal{D}_P : J^k FP \to EP, \mathcal{D}_P(J^k_x s) = D_P s(x)$. By definition of the k-th order gauge natural operator we see that each $\mathcal{D}_P$ is well defined and smooth. We must prove that the following diagram:

$$\begin{array}{ccc} J^k FP & \xrightarrow{\mathcal{D}_P} & EP \\ {\scriptstyle J^k Ff}\downarrow & & \downarrow{\scriptstyle Ef} \\ J^k F\bar{P} & \xrightarrow{\mathcal{D}_{\bar{P}}} & E\bar{P} \end{array} \tag{2.7}$$

commutes for all $\mathcal{PB}_n(G)$-morphism $f : P \to \bar{P}$. In fact $\mathcal{D}_{\bar{P}} \circ J^k Ff(J^k_x s) = D_{\bar{P}}(Ff \circ s \circ BFf^{-1})(BFf(x)) = Ef \circ D_P s \circ Bf^{-1}(Bf(x)) = Ef \circ D_P s(x) = Ef \circ \mathcal{D}_P(J^k_x s)$. (Every $\mathcal{PB}_n(G)$-morphism is a local isomorphism and D_P depends only on $\mathrm{germ}_x s$, so we can append restrictions, if necessary.)

Conversely, for every natural transformation $\mathcal{D} : J^k F \to E$ we define the k-th order gauge natural operator by $D_P s(x) = \mathcal{D}_P(J^k_x s)$. Because every $\mathcal{D}_P$ is a base preserving $\mathcal{FM}$-morphism (Lemma 2.10) we see that $\mathcal{D}_P \circ J^k s$ is a section, evidently smooth. From $D_{\bar{P}}(Ff \circ s \circ Bf^{-1})(Bf(x)) = \mathcal{D}_{\bar{P}}(J^k_{Bf(x)}(Ff \circ s \circ Bf^{-1})) = \mathcal{D}_{\bar{P}} \circ J^k Ff(J^k_x s) = Ef \circ \mathcal{D}_P(J^k_x s) = Ef \circ D_P s(x)$ we get the first condition on the gauge natural operator, the second can be seen at once. The order is clear from definition.

$\diamondsuit$

We denote the standard fibers of FP and EP by $F_0 = F_0(\mathbb{R}^n \times G)$ and $E_0 = E_0(\mathbb{R}^n \times G)$. To each $W^r_n G$-equivariant map $f : F_0 \to E_0$ we can associate the $\mathcal{FM}$-morphism f_P which is given by $f_P([\zeta, u]) = [\zeta, f(u)]$.

Lemma 2.10. *Let $\Phi : FP \to EP$ be a $\mathcal{FM}$-morphism, then $Eg \circ \Phi = \Phi \circ Fg$ holds for each $\mathcal{PB}_n(G)$-morphism $g : P \to P$ iff there exists a unique $W^r_n G$-equivariant map $f : F_0 \to E_0$ such that $f_P = \Phi$.*

Proof: Denote by p_P (resp. p_{FP}, resp. p_{EP}) the projection of the bundle $W^r P$ (resp. FP, resp. EP). From $p_{EP} \circ Eg \circ \Phi = p_{EP} \circ \Phi \circ Fg$ we get $g_0 \circ \Phi_0 \circ p_{FP} = \Phi_0 \circ g_0 \circ p_{FP}$ so $g_0 \circ \Phi_0 = \Phi_0 \circ g_0$ for all $\mathcal{PB}_n(G)$-morphism $g : P \to P$ and so $\Phi_0 = \mathrm{id}_X$. We define a map $\Phi_\zeta : F_0 \to E_0$ by the relation

$\Phi([\zeta, u]) = [\zeta, \Phi_\zeta(u)]$ for any $x \in X$ and $\zeta \in p_P^{-1}(x)$. Φ_ζ is independent of ζ over x. In fact, $W^r g(\zeta) = \zeta \cdot a$ for $\zeta = J^r_{(0,e)}\psi$, $a = J^r_{(0,e)}\phi$ and $g = \psi \circ \phi \circ \psi^{-1}$ and so $[W^r g(\zeta), \Phi_\zeta(u)] = Eg \circ \Phi([\zeta, u]) = \Phi \circ Fg([\zeta, u]) = [W^r g(\zeta), \Phi_{W^r g(\zeta)}(u)] = [W^r g(\zeta), \Phi_{\zeta \cdot a}(u)]$, which implies $\Phi_\zeta = \Phi_{\zeta \cdot a}$. By transitivity of the action of $W^r_n G$ on $W^r P$ we get the independence of Φ_ζ from the choice of ζ in the fiber over x. Moreover, Φ_ζ is independent of x. Choose a $\mathcal{PB}_n(G)$-morphism g sending ζ_1 over x_1 to ζ_2 over x_2 then $[W^r g(\zeta_1), \Phi_{\zeta_1}(u)] = Eg \circ \Phi([\zeta_1, u]) = \Phi \circ Fg([\zeta_1, u]) = [W^r g(\zeta_1), \Phi_{\zeta_2}(u)]$, which implies $\Phi_{\zeta_1}(u) = \Phi_{\zeta_2}(u)$. Now we put $f = \Phi_\zeta$. From $[\zeta, f(u)] = [\zeta, \Phi_\zeta(u)] = \Phi([\zeta, u]) = \Phi([\zeta \cdot a^{-1}, a \cdot u]) = [\zeta \cdot a^{-1}, \Phi_\zeta(a \cdot u)] = [\zeta, a^{-1} \cdot f(a \cdot u)]$ for all $a \in W^r_n G$ we see that $f(a \cdot u) = a \cdot f(u)$ and so we get the $W^r_n G$-equivariant map $f : F_0 \to E_0$ such that $[\zeta, f(u)] = \Phi([\zeta, u])$. The uniqueness and the converse implication are obvious.

$\diamondsuit$

Theorem 2.11. *Natural transformations $F \to E$ between two r-th order gauge natural bundle functors over n-dimensional manifolds are in canonical bijection with the $W^r_n G$-equivariant maps $F_0 \to E_0$ between the standard fibers.*

Proof: To each $W^r_n G$-equivariant map $f : F_0 \to E_0$ we can associate the natural transformation $D_f : F \to E$ defined by $D_f : P \mapsto f_P$ for each bundle $P \in \mathrm{Ob}(\mathcal{PB}_n(G))$. From the equation $f_{\bar{P}} \circ Fg([\zeta, u]) = f_{\bar{P}}([W^r g(\zeta), u]) = [W^r g(\zeta), f(u)] = Eg([\zeta, f(u)]) = Eg \circ f_P([\zeta, u])$ for all $\mathcal{PB}_n(G)$-morphisms $g : P \to \bar{P}$ we see that D_f is really a natural transformation.

Now we shall show that the correspondence $f \mapsto D_f$ is bijective. $D_{f_1} = D_{f_2}$ implies $f_{1P} = f_{2P}$ for all $P \in \mathrm{Ob}(\mathcal{PB}_n(G))$, by Lemma 2.10 we get $f_1 = f_2$ and so the correspondence is injective. For an arbitrary natural transformation $D : F \to E$ and for fixed P there must exist by Lemma 2.10 a unique $W^r_n G$-equivariant map $f : F_0 \to E_0$ such that $f_P = D_P$. This f is independent of P. In fact, suppose that $f_{P_1} = D_{P_1}$ and $\bar{f}_{P_2} = D_{P_2}$ for some $P_1, P_2 \in \mathrm{Ob}(\mathcal{PB}_n(G))$, then $\bar{f}_{P_2} \circ Fg = Eg \circ f_{P_1}$ for any $\mathcal{PB}_n(G)$-morphism $g : P_1 \to P_2$, which implies $[W^r g(\zeta), \bar{f}(u)] = \bar{f}_{P_2} \circ Fg([\zeta, u]) = Eg \circ f_{P_1}([\zeta, u]) = [W^r g(\zeta), f(u)]$ and so $f = \bar{f}$, thus the correspondence is surjective.

$\diamondsuit$

Chapter 3

Variational Theory on Fibered Manifolds

Our main goal in this chapter is to discuss properties and underlying geometric structures needed in the general variational theory. We focus our attention on the concepts which are necessary in the Einstein-Yang-Mills theory. We prefer the notion of differential forms for a Lagrangian defining a global variational functional. A global characterization of extremals in terms of partial differential equations is achieved with the help of the so called Lepage forms, allowing us to express the variational derivatives in a coordinate-independent form. We define a gauge natural structure of a gauge natural field theories. We also discuss Noether's theorem and the first variation formula for the so called induced variations for a gauge natural Lagrangian.

3.1 The Lagrangian and the Action Function

Let $\pi : Y \to X$ be a fibered manifold with $\dim Y = n + m$ over a n-dimensional orientable base manifold X. If $W \subset Y$ is an open set, we denote by $\Omega_0^r W$ the ring of functions on $W^r = (\pi^{r,0})^{-1}(W)$; we denote by $\Omega_p^r W$ the $\Omega_0^r W$-module of differential p-forms on W^r and the exterior algebra of forms on W^r is denoted by $\Omega^r W$. A differential p-form ρ on Y is said to be **π-horizontal** (or simply **horizontal**), if for each point $y \in Y$ the contraction $i_\xi \rho(y)$ vanishes whenever ξ is a vertical vector. The module of $\pi^{r,0}$-horizontal (resp. π^r-horizontal, where $\pi^r = \alpha$ is the source projection) p-forms on W^r is denoted by $\Omega_{p,Y}^r W$ (resp. $\Omega_{p,X}^r W$).

A **Lagrangian of order** r for a fibered manifold Y is a π^r-horizontal n-form λ on the open set W^r of the r-jet prolongation J^rY of Y, i.e. $\lambda \in \Omega^r_{n,X}W$. In a fiber chart (V,ψ), $\psi = (x^i, y^\sigma)$ on Y a Lagrangian of order r defined on V^r can be expressed as $\lambda = \mathcal{L}\omega_0$, where $\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ and $\mathcal{L} : V^r \to \mathbb{R}$ is the so called **Lagrange function** associated with (V,ψ). From this coordinate representation we see that a Lagrangian of order r can be considered as a morphism $\tilde{\lambda}$ of fibered manifolds:

$$\begin{tikzcd} J^rY \arrow[r, "\tilde{\lambda}"] \arrow[d, "\pi^r"'] & \wedge^n T^*X \arrow[dl] \\ X & \end{tikzcd}$$

But we shall prefer the notion of horizontal n-forms.

Let $\Omega \subset \pi(W)$ be a compact, n-dimensional submanifold of X with boundary. We denote by $\Gamma_{\Omega,W}(Y)$ the set of smooth sections of Y from Ω into W. For a Lagrangian $\lambda \in \Omega^r_{n,X}W$ we define the so called **action function** (or the **variational function**) $\lambda_\Omega : \Gamma_{\Omega,W}(Y) \to \mathbb{R}$ by $\lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda$. If ξ is a projectable vector field on an open set W, i.e. if there exists a vector field ξ_0 on $\pi(W)$ such that ξ and ξ_0 are π-related ($T\pi \circ \xi = \xi_0 \circ \pi$), then we have for their flows $\pi \circ \mathrm{Fl}^\xi_t = \mathrm{Fl}^{\xi_0}_t \circ \pi$. We define the **variation** (or the **deformation**) of the section $\gamma \in \Gamma_{\Omega,W}(Y)$ induced by the vector field ξ by $\gamma_t = \mathrm{Fl}^\xi_t \circ \gamma \circ (\mathrm{Fl}^{\xi_0}_t)^{-1}$. From the computation on the domain of γ_t

$$\begin{aligned} \pi \circ \gamma_t = \pi \circ \mathrm{Fl}^\xi_t \circ \gamma \circ (\mathrm{Fl}^{\xi_0}_t)^{-1} &= \mathrm{Fl}^{\xi_0}_t \circ \pi \circ \gamma \circ (\mathrm{Fl}^{\xi_0}_t)^{-1} \\ &= \mathrm{Fl}^{\xi_0}_t \circ (\mathrm{Fl}^{\xi_0}_t)^{-1} = \mathrm{id} \end{aligned}$$

we see that the variation is a 1-parameter family of sections of Y. We say that a section $\gamma \in \Gamma_{\Omega,W}(Y)$ is a **stable point** of the variational function λ_Ω with respect to its variation induced by the vector field ξ, if

$$\left(\frac{d}{dt}\lambda_{\mathrm{Fl}^{\xi_0}_t(\Omega)}(\gamma_t)\right)_0 = 0. \tag{3.1}$$

For a projectable vector field ξ on Y we can define a vector field $J^r\xi$ on J^rY - the so called r-**jet prolongation** of ξ by $J^r\xi(J^r_x\gamma) = \frac{d}{dt}\big|_0 J^r(\mathrm{Fl}^\xi_t)(J^r_x\gamma)$ for each $J^r_x\gamma$ belonging to the domain of $J^r(\mathrm{Fl}^\xi_t)$. Then using the Lie derivative ∂, Equation (3.1) is equivalent to

$$(\partial_{J^r\xi}\lambda)_\Omega(\gamma) = 0. \tag{3.2}$$

In fact, this follows from the computation

$$\begin{aligned}
\left(\frac{d}{dt}\lambda_{\mathrm{Fl}_t^{\xi_0}(\Omega)}(\gamma_t)\right)_0 &= \left(\frac{d}{dt}\int_{\mathrm{Fl}_t^{\xi_0}(\Omega)} (J^r(\mathrm{Fl}_t^{\xi}\circ\gamma\circ(\mathrm{Fl}_t^{\xi_0})^{-1}))^*\lambda\right)_0 \\
&= \left(\frac{d}{dt}\int_{\mathrm{Fl}_t^{\xi_0}(\Omega)} (J^r\mathrm{Fl}_t^{\xi}\circ J^r\gamma\circ(\mathrm{Fl}_t^{\xi_0})^{-1}))^*\lambda\right)_0 \\
&= \left(\frac{d}{dt}\int_{\mathrm{Fl}_t^{\xi_0}(\Omega)} (\mathrm{Fl}_t^{\xi_0})^{-1*}\circ J^r\gamma^*\circ(J^r\mathrm{Fl}_t^{\xi})^*\lambda\right)_0 \\
&= \left(\frac{d}{dt}\int_{\Omega} J^r\gamma^*\circ(J^r\mathrm{Fl}_t^{\xi})^*\lambda\right)_0 = \int_\Omega J^r\gamma^*\left(\frac{d}{dt}(J^r\mathrm{Fl}_t^{\xi})^*\lambda\right)_0 \\
&= \int_\Omega J^r\gamma^*\left(\frac{d}{dt}(\mathrm{Fl}_t^{J^r\xi})^*\lambda\right)_0 = \int_\Omega J^r\gamma^*\partial_{J^r\xi}\lambda = (\partial_{J^r\xi}\lambda)_\Omega(\gamma).
\end{aligned}$$

We call the variational function $(\partial_{J^r\xi}\lambda)_\Omega : \Gamma_{\Omega,W}(Y)\to\mathbb{R}$ (associated with the Lagrangian $\partial_{J^r\xi}\lambda$) the **variational derivative** or the **first variation** of the variational function λ_Ω by the vector field ξ.

Let ρ be a differential k-form on J^rY. Then there exists one and only one π^r-horizontal k-form $h\rho$ on $J^{r+1}Y$ such that

$$J^r\gamma^*\rho = J^{r+1}\gamma^*h\rho \tag{3.3}$$

for all sections γ of Y. To prove the existence we set $h\rho(J_x^{r+1}\gamma) = \pi^{r+1*}\circ J^r\gamma^*\rho(J_x^{r+1}\gamma)$ which satisfies Equation (3.3). The uniqueness for $k=0$ and $k\ge n$ is evident and for $1\le k\le n$ in a fiber chart (V,ψ), $\psi=(x^i,y^\sigma)$ on Y we can write $h\rho$ as $\rho_{i_1i_2\dots i_k}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}$, so the condition $J^{r+1}\gamma^*h\rho=0$ for all γ implies $\rho_{i_1i_2\dots i_k}=0$, thus $h\rho$ is unique. The mapping $\Omega_k^rW\ni\rho\to h\rho\in\Omega_k^{r+1}W$ is called the **horizontalization**. The horizontalization considered as a morphism of exterior algebras $\Omega^rW\ni\rho\to h\rho\in\Omega^{r+1}W$ is a unique $\mathbb{R}$-linear, exterior product preserving mapping such that for any function $f:W^r\to\mathbb{R}$ and any fiber chart (V,ψ), $\psi=(x^i,y^\sigma)$ in W

$$hf = f\circ\pi^{r+1,r},\ h(df) = d_if\,dx^i,\ d_if = \frac{\partial f}{\partial x^i} + \sum_{j_1\le j_2\le\cdots\le j_k}\frac{\partial f}{\partial y^\sigma_{j_1j_2\dots j_k}}y^\sigma_{j_1j_2\dots j_ki}, \tag{3.4}$$

where $0\le k\le r$. This follows directly from the definitions and the uniqueness follows from

$$hdx^i = dx^i,\ hdy^\sigma_{j_1j_2\dots j_k} = y^\sigma_{j_1j_2\dots j_ki}dx^i,$$

which we obtained from (3.4). The function $d_if : V^{r+1}\to\mathbb{R}$ is called the i-th **formal derivative** of f with respect to the fiber chart (V,ψ).

Similarly we define the **horizontalization** of tangent vectors as a vector bundle morphism $h:TJ^{r+1}Y\to TJ^rY$ over $\pi^{r+1,r}$ by the formula $h\xi = T_xJ^r\gamma\circ$

$T\pi^{r+1} \cdot \xi$ for $\xi \in T_{J^{r+1}_x\gamma}J^{r+1}Y$. We call $h\xi$ the **horizontal component** of ξ and $p\xi = T\pi^{r+1,r} \cdot \xi - h\xi$ the **contact component** of ξ. For $\rho \in \Omega^r_q W$ and vectors $\xi_1, \xi_2, \ldots, \xi_q$ tangent to $J^{r+1}Y$ at a point $J^{r+1}_x\gamma \in W^{r+1}$ we define the k**-contact component** $p_k\rho$ of the form ρ by

$$p_k\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_q) = \sum_{j_1<j_2<\cdots<j_k} \sum_{j_{k+1}<j_{k+2}<\cdots<j_q}$$
$$\epsilon^{j_1 j_2 \cdots j_k j_{k+1} j_{k+2} \cdots j_q} \rho(J^r_x\gamma)(p\xi_{j_1}, p\xi_{j_2}, \ldots, p\xi_{j_k}, h\xi_{j_{k+1}}, \ldots, h\xi_{j_q}).$$

It is convenient to write $h\rho = p_0\rho$ and $p\rho = \sum_{i=1}^q p_i\rho$ and extend the definition to functions; for $f : W^r \to \mathbb{R}$ we define $hf = \pi^{r+1,r*}f$ and $pf = 0$. Now we have the **canonical decomposition** $\pi^{r+1,r*}\rho = h\rho + p\rho$ into the **horizontal component** $h\rho$ of ρ, which agrees with the horizontalization defined before, and into the **contact component** $p\rho$ of ρ. A q-form $\rho \in \Omega^r_q W$ is called **contact**, if $h\rho = 0$, and k**-contact**, if $\pi^{r+1,r*}\rho = p_k\rho$. From

$$h\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \ldots, \xi_q) = (J^r\gamma^*\rho)(x)(T\pi^{r+1} \cdot \xi_1, T\pi^{r+1} \cdot \xi_2, \ldots, T\pi^{r+1} \cdot \xi_q)$$

we see that ρ is contact if and only if $J^r\gamma^*\rho = 0$ for every smooth section γ of Y defined on an open subset of W. This implies that contact forms form an ideal in $\Omega^r W$ - the so called **contact ideal**. Further it can be proved that ρ is horizontal if and only if $p\rho = 0$ and that the forms $p_1\rho, \ldots, p_q\rho$ are contact (see [30]). If (V, ψ), $\psi = (x^i, y^\sigma)$ is a fiber chart on Y, then the forms

$$dx^i,\ \eta^\sigma_J = dy^\sigma_J - y^\sigma_{Jj}dx^j,\ dy^\sigma_I,$$

for multiindices $0 \le |J| \le r-1$ and $|I| = r$ form a basis of linear forms on the set V^r, furthermore the 1-forms η^σ_J are contact.

A differential form $\Theta \in \Omega^s_n W$ is called **Lepage form**, if for each $\pi^{s,0}$-vertical vector field ξ on W^s we have $hi_\xi d\Theta = 0$. A Lepage form Θ_λ is called a **Lepage equivalent** of a Lagrangian $\lambda \in \Omega^r_{n,X}W$ if $h\Theta_\lambda = \lambda$ (possibly up to a jet projection, i.e. we denote a form on some jet prolongation and its pullback by a jet projection by the same symbol). If we define the variational function $\Theta_{\lambda\Omega} : \Gamma_{\Omega,W}(Y) \to \mathbb{R}$ for a Lepage equivalent of a Lagrangian λ by the same rule as before for a Lagrangian, then their variational functions are the same. In fact, for a Lepage equivalent $\Theta_\lambda \in \Omega^{r-1}_n W$ we have

$$\Theta_{\lambda\Omega}(\gamma) = \int_\Omega J^{r-1}\gamma^*\Theta_\lambda = \int_\Omega J^r\gamma^* h\Theta_\lambda = \int_\Omega J^r\gamma^*\lambda = \lambda_\Omega(\gamma).$$

Theorem 3.1. *For each Lagrangian $\lambda \in \Omega^2_{n,X}W$ there exists a Lepage equivalent $\Theta_\lambda \in \Omega^3_n W$ such that in any fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$ $(V \subset W)$ it has the form*

$$\Theta_\lambda = \mathcal{L}\omega_0 + \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_i} - d_p \frac{\partial \mathcal{L}}{\partial y^\sigma_{pi}}\right)\eta^\sigma \wedge \omega_i + \frac{\partial \mathcal{L}}{\partial y^\sigma_{ji}}\eta^\sigma_j \wedge \omega_i, \qquad (3.5)$$

where $\lambda = \mathcal{L}\omega_0$ and $\omega_i = i_{\frac{\partial}{\partial x^i}}\omega_0$ is a contraction of ω_0.

Proof: We search for a form Θ_λ with undetermined coefficients f^i_σ , f^{ij}_σ, such that in any fiber chart $\Theta_\lambda = \mathcal{L}\omega_0 + (f^i_\sigma \eta^\sigma + f^{ij}_\sigma \eta^\sigma_j) \wedge \omega_i$. If we consider different fiber coordinates $(\bar{x}^i, \bar{y}^\sigma)$, then in the obvious shorthand notation we have the following transformation rules for the jet coordinates

$$\bar{y}^\sigma_i = \left(\frac{\partial \bar{y}^\sigma}{\partial x^j} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_j\right) \frac{\partial x^j}{\partial \bar{x}^i}$$

$$\bar{y}^\sigma_{i_1 i_2} = \frac{\partial^2 \bar{y}^\sigma}{\partial x^{j_1} \partial x^{j_2}} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\nu \partial x^{j_2}} y^\nu_{j_1} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{j_1 j_2} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} +$$
$$\left(\frac{\partial \bar{y}^\sigma}{\partial x^{j_1}} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{j_1}\right) \frac{\partial^2 x^{j_1}}{\partial \bar{x}^{i_1} \partial \bar{x}^{i_2}} + \frac{\partial^2 \bar{y}^\sigma}{\partial x^{j_1} \partial y^\nu} y^\nu_{j_2} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\nu \partial y^\rho} y^\nu_{j_1} y^\rho_{j_2} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}}.$$

Thus for the contact forms we get

$$\bar{\eta}^\sigma = d\bar{y}^\sigma - \bar{y}^\sigma_j d\bar{x}^j = \frac{\partial \bar{y}^\sigma}{\partial x^j} dx^j + \frac{\partial \bar{y}^\sigma}{\partial y^\rho} dy^\rho - \left(\frac{\partial \bar{y}^\sigma}{\partial x^k} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_k\right) \frac{\partial x^k}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^i} dx^i$$
$$= \frac{\partial \bar{y}^\sigma}{\partial x^j} dx^j + \frac{\partial \bar{y}^\sigma}{\partial y^\rho} dy^\rho - \left(\frac{\partial \bar{y}^\sigma}{\partial x^i} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_i\right) dx^i = \frac{\partial \bar{y}^\sigma}{\partial y^\nu} (dy^\nu - y^\nu_i dx^i) = \frac{\partial \bar{y}^\sigma}{\partial y^\nu} \eta^\nu$$

$$\bar{\eta}^\sigma_j = d\bar{y}^\sigma_j - \bar{y}^\sigma_{ji} d\bar{x}^i = \left[\left(\frac{\partial \bar{y}^\sigma}{\partial x^{j_1} \partial x^k} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu \partial x^k} y^\nu_{j_1}\right) \frac{\partial x^{j_1}}{\partial \bar{x}^j}\right.$$
$$\left. + \left(\frac{\partial \bar{y}^\sigma}{\partial x^{j_1}} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{j_1}\right) \frac{\partial^2 x^{j_1}}{\partial \bar{x}^j \partial x^k}\right] dx^k + \left[\left(\frac{\partial \bar{y}^\sigma}{\partial x^{j_1} \partial y^\rho} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu \partial y^\rho} y^\nu_{j_1}\right) \frac{\partial x^{j_1}}{\partial \bar{x}^j}\right] dy^\rho$$
$$+ \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \frac{\partial x^k}{\partial \bar{x}^j} dy^\rho_k - \left[\frac{\partial^2 \bar{y}^\sigma}{\partial x^{j_1} \partial x^k} \frac{\partial x^{j_1}}{\partial \bar{x}^j} + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\nu \partial x^k} y^\nu_{j_1} \frac{\partial x^{j_1}}{\partial \bar{x}^j} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{j_1 k} \frac{\partial x^{j_1}}{\partial \bar{x}^j}\right.$$
$$\left. + \left(\frac{\partial \bar{y}^\sigma}{\partial x^{j_1}} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu} y^\nu_{j_1}\right) \frac{\partial^2 x^{j_1}}{\partial \bar{x}^j \partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^k} + \frac{\partial^2 \bar{y}^\sigma}{\partial x^{j_1} \partial y^\nu} y^\nu_k \frac{\partial x^{j_1}}{\partial \bar{x}^j} + \frac{\partial^2 \bar{y}^\sigma}{\partial y^\nu \partial y^\rho} y^\nu_{j_1} y^\rho_k \frac{\partial x^{j_1}}{\partial \bar{x}^j}\right] dx^k$$
$$= \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \frac{\partial x^k}{\partial \bar{x}^j} (dy^\rho_k - y^\rho_{kl} dx^l) + \left(\frac{\partial \bar{y}^\sigma}{\partial x^{j_1} \partial y^\rho} + \frac{\partial \bar{y}^\sigma}{\partial y^\nu \partial y^\rho} y^\nu_{j_1}\right) \frac{\partial x^{j_1}}{\partial \bar{x}^j} (dy^\rho - y^\rho_k dx^k)$$
$$= \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \frac{\partial x^k}{\partial \bar{x}^j} \eta^\rho_k + d_{j_1} \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \frac{\partial x^{j_1}}{\partial \bar{x}^j} \eta^\rho = \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \frac{\partial x^k}{\partial \bar{x}^j} \eta^\rho_k + \bar{d}_j \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \eta^\rho,$$

where in the last equality we used the fact that if $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$ is another fiber chart on Y such that $V \cap \bar{V} \neq \emptyset$ and $\bar{d}_i$ the i-th formal derivative with respect to this fiber chart, then for any function $f : V \cap \bar{V} \to \mathbb{R}$ we have $\bar{d}_i f = d_j f \frac{\partial x^j}{\partial \bar{x}^i}$.

Now we can split $f^{ij}_\sigma = f^{(ij)}_\sigma + f^{[ij]}_\sigma$ into a symmetric $f^{(ij)}_\sigma$ and an antisymmetric $f^{[ij]}_\sigma$ part. But using the transformation rule for $\bar{\eta}^\sigma_j$ and the transformation rule $\bar{\omega}_i = \frac{\partial x^j}{\partial \bar{x}^i} J \omega_j$ where $J = \det \frac{\partial \bar{x}^j}{\partial x^i}$ is a Jacobian we see that only the symmetric term $\frac{\partial \bar{y}^\sigma}{\partial y^\rho} \frac{\partial x^k}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^i} J(\eta^\rho_k \wedge \omega_l + \eta^\rho_l \wedge \omega_k)$ contributes to the transformation of the symmetric term $\bar{\eta}^\rho_k \wedge \bar{\omega}_l + \bar{\eta}^\rho_l \wedge \bar{\omega}_k$ and similarly for the antisymmetric terms.

Thus we can take the antisymmetric part $f_\sigma^{[ij]}$ to be zero and we assume that $f_\sigma^{ij} = f_\sigma^{ji}$. This condition and what follows prove the global existence of Θ_λ. The condition $h\Theta_\lambda = \lambda$ is satisfied. For any vector field

$$\xi = \xi^i \frac{\partial}{\partial x^i} + \xi^\sigma \frac{\partial}{\partial y^\sigma} + \xi_j^\sigma \frac{\partial}{\partial y_j^\sigma} + \xi_{jk}^\sigma \frac{\partial}{\partial y_{jk}^\sigma} \tag{3.6}$$

we compute what the condition on Θ_λ to be a Lepage form requires (using the fact that $dx^j \wedge \omega_i = \delta_i^j \omega_0$):

$$\begin{aligned}
hi_\xi d\Theta_\lambda &= hi_\xi (d\mathcal{L} \wedge \omega_0 + (df_\sigma^i \wedge \eta^\sigma + df_\sigma^{ij} \wedge \eta_j^\sigma) \wedge \omega_i - (f_\sigma^i \eta_i^\sigma + f_\sigma^{ij} \eta_{ij}^\sigma) \wedge \omega_0) \\
&= (i_\xi d\mathcal{L} - \xi^i d_i \mathcal{L} - i_\xi \eta^\sigma d_i f_\sigma^i - i_\xi \eta_j^\sigma d_i f_\sigma^{ij} - f_\sigma^i i_\xi \eta_i^\sigma - f_\sigma^{ij} i_\xi \eta_{ij}^\sigma)\omega_0 \\
&= \Bigg(\frac{\partial \mathcal{L}}{\partial y^\sigma}(\xi^\sigma - y_j^\sigma \xi^j) + \frac{\partial \mathcal{L}}{\partial y_i^\sigma}(\xi_i^\sigma - y_{ik}^\sigma \xi^k) + \frac{\partial \mathcal{L}}{\partial y_{ij}^\sigma}(\xi_{ij}^\sigma - y_{ijk}^\sigma \xi^k) \\
&\quad -(\xi^\sigma - y_j^\sigma \xi^j) d_i f_\sigma^i - (\xi_j^\sigma - y_{jk}^\sigma \xi^k) d_i f_\sigma^{ij} - f_\sigma^i (\xi_i^\sigma - y_{ik}^\sigma \xi^k) - f_\sigma^{ij} (\xi_{ij}^\sigma - y_{ijk}^\sigma \xi^k) \Bigg) \omega_0 \\
&= \Bigg(\left(\frac{\partial \mathcal{L}}{\partial y^\sigma} - d_i f_\sigma^i \right)(\xi^\sigma - y_j^\sigma \xi^j) + \left(\frac{\partial \mathcal{L}}{\partial y_i^\sigma} - d_j f_\sigma^{ji} - f_\sigma^i \right)(\xi_i^\sigma - y_{ik}^\sigma \xi^k) \\
&\quad + \left(\frac{\partial \mathcal{L}}{\partial y_{ij}^\sigma} - f_\sigma^{ij} \right)(\xi_{ij}^\sigma - y_{ijk}^\sigma \xi^k) \Bigg) \omega_0.
\end{aligned}$$

Thus we get

$$f_\sigma^{ij} = \frac{\partial \mathcal{L}}{\partial y_{ij}^\sigma}, \; f_\sigma^i = \frac{\partial \mathcal{L}}{\partial y_i^\sigma} - d_p \frac{\partial \mathcal{L}}{\partial y_{pi}^\sigma} \tag{3.7}$$

and we obtain Equation (3.5).

◇

The Lepage equivalent from Theorem 3.1 is called the **principal Lepage equivalent** and it generalizes the Poincaré-Cartan form $\Theta_\lambda = \mathcal{L}\omega_0 + \frac{\partial \mathcal{L}}{\partial y_i^\sigma} \eta^\sigma \wedge \omega_i$ of a first order Lagrangian $\lambda \in \Omega^1_{n,X} W$ expressed in a fiber chart by $\lambda = \mathcal{L}\omega_0$. For higher order generalizations see [30]. It can be shown that the principal Lepage equivalent of the Hilbert Lagrangian is of first order (see [35]).

For a Lepage equivalent $\Theta_\lambda \in \Omega_n^{r-1} W$ of a Lagrangian $\lambda \in \Omega^r_{n,X} W$ the Lie derivative $\partial_{J^r\xi}\lambda$ can be expressed by the **first variation formula**

$$\partial_{J^r\xi}\lambda = hi_{J^{r-1}\xi} d\Theta_\lambda + hdi_{J^{r-1}\xi}\Theta_\lambda. \tag{3.8}$$

In fact, we have

$$\partial_{J^r\xi}\lambda = \partial_{J^r\xi}(h\Theta_\lambda) = h\partial_{J^{r-1}\xi}\Theta_\lambda = hi_{J^{r-1}\xi} d\Theta_\lambda + hdi_{J^{r-1}\xi}\Theta_\lambda.$$

Thus for the first variation of the variational function λ_Ω by the vector field ξ we get

$$\begin{aligned}(\partial_{J^r\xi}\lambda)_\Omega(\gamma) &= \int_\Omega J^r\gamma^*\partial_{J^r\xi}\lambda = \int_\Omega J^r\gamma^*(hi_{J^{r-1}\xi}d\Theta_\lambda + hdi_{J^{r-1}\xi}\Theta_\lambda) \\ &= \int_\Omega J^{r-1}\gamma^* i_{J^{r-1}\xi}d\Theta_\lambda + \int_\Omega J^{r-1}\gamma^* di_{J^{r-1}\xi}\Theta_\lambda \\ &= \int_\Omega J^{r-1}\gamma^* i_{J^{r-1}\xi}d\Theta_\lambda + \int_{\partial\Omega} J^{r-1}\gamma^* i_{J^{r-1}\xi}\Theta_\lambda.\end{aligned}$$

This is called the **integral first variation formula**. It shows the role of Lepage forms in deriving such a decomposition of the n-form $\partial_{J^r\xi}\lambda$ into two terms, the first of which depends only on ξ and on the Lagrangian and it corresponds to the Euler-Lagrange expressions, and the second one only on the values of $J^{r-1}\xi$ on the boundary $\partial\Omega$ of Ω and on the choice of the Lepage equivalent.

Theorem 3.2. *For the principal Lepage equivalent from Theorem 3.1 the Euler-Lagrange term has a chart expression*

$$hi_\xi d\Theta_\lambda = \left(\frac{\partial\mathcal{L}}{\partial y^\sigma} - d_i\left(\frac{\partial\mathcal{L}}{\partial y^\sigma_i} - d_p\frac{\partial\mathcal{L}}{\partial y^\sigma_{pi}}\right)\right)(\xi^\sigma - y^\sigma_j\xi^j)\omega_0$$

(ξ is as in Equation (3.6)) and the boundary term has a chart expression

$$hdi_\xi\Theta_\lambda = d_i\left(\mathcal{L}\xi^i + \left(\frac{\partial\mathcal{L}}{\partial y^\sigma_i} - d_p\frac{\partial\mathcal{L}}{\partial y^\sigma_{pi}}\right)(\xi^\sigma - y^\sigma_k\xi^k) + \frac{\partial\mathcal{L}}{\partial y^\sigma_{ij}}(\xi^\sigma_j - y^\sigma_{jk}\xi^k)\right)\omega_0.$$

Proof: The Euler-Lagrange term is found immediately from the computation in the proof of Theorem 3.1. Further we have

$$\begin{aligned}hdi_\xi\Theta_\lambda &= hd(\mathcal{L}\xi^i\omega_i + (f^i_\sigma(\xi^\sigma - y^\sigma_k\xi^k) + f^{ij}_\sigma(\xi^\sigma_j - y^\sigma_{jk}\xi^k)) \wedge \omega_i \\ &\quad -(f^i_\sigma\eta^\sigma + f^{ij}_\sigma\eta^\sigma_j) \wedge i_\xi\omega_i) \\ &= d_i(\mathcal{L}\xi^i + f^i_\sigma(\xi^\sigma - y^\sigma_k\xi^k) + f^{ij}_\sigma(\xi^\sigma_j - y^\sigma_{jk}\xi^k))\omega_0.\end{aligned}$$

Using Equations (3.7) finishes the proof.

◇

3.2 Gauge Natural Structures

Many physical theories can be described as a gauge natural field theory, i.e. they have the following gauge natural structure. A **gauge natural structure** is made of the following items:

1. a **structure bundle** P which is a principal bundle over an n-dimensional manifold X with a Lie group G,

2. a **configuration bundle** C which is a gauge natural bundle of order (s,r) which is associated to $W^{s,r}P$, [1]

3. a Lagrangian λ of order r on C which is **gauge natural**, i.e. λ is a r-th order gauge natural operator from C to $\wedge^n T^*B$.

We show that condition 3 can be replaced by the equivalent one.

Theorem 3.3. *A Lagrangian λ of order 2 on C is gauge natural iff Θ_λ is* $\mathrm{Aut}(P)$*-invariant, i.e. $J^3Cf^*\Theta_\lambda = \Theta_\lambda$ for all local automorphisms $f \in \mathrm{Aut}(P)$.*

Proof: First we prove that $(J^rCf)^*\lambda = \lambda$ holds iff $\wedge^n T^*Bf \circ \tilde{\lambda} = \tilde{\lambda} \circ J^rCf$, where we denote for the moment by λ the Lagrangian considered as a horizontal form and by $\tilde{\lambda}$ the Lagrangian considered as morphism. We note that the functors C and $\wedge^n T^*B$ are considered to be restricted to the subcategory of $\mathcal{PB}_n(G)$ with a fixed principal bundle P. Let $u = J^r_x\gamma \in J^rC$ and $\xi_1, \dots, \xi_n \in T_uJ^rC$, then $\tilde{\lambda}$ and λ are related by $\lambda(u)(\xi_1,\dots,\xi_n) = \tilde{\lambda}(u)(T_u\pi^r(\xi_1),\dots,T_u\pi^r(\xi_n))$. Now the statement follows from the computation below (π^r is a submersion)

$$\begin{aligned}
(J^rCf)^*\lambda(u)(\xi_1,\dots,\xi_n) &= \lambda(J^rCf(u))(T_uJ^rCf(\xi_1),\dots,T_uJ^rCf(\xi_n)) \\
&= \tilde{\lambda} \circ J^rCf(u)(T_u(\pi^r \circ J^rCf)(\xi_1),\dots,T_u(\pi^r \circ J^rCf)(\xi_n)) \\
&= \tilde{\lambda} \circ J^rCf(u)(T_xBf \circ T_u\pi^r(\xi_1),\dots,T_xBf \circ T_u\pi^r(\xi_n)) \\
&= \wedge^n T^*Bf^{-1} \circ \tilde{\lambda} \circ J^rCf(u)(T_u\pi^r(\xi_1),\dots,T_u\pi^r(\xi_n)).
\end{aligned}$$

Using Theorem 2.9 we have the following diagram:

$$\begin{tikzcd}
J^rCP \arrow[r, "\tilde{\lambda}"] \arrow[d, "J^rCf"'] & \wedge^n T^*BP \arrow[d, "\wedge^n T^*Bf"] \\
J^rCP \arrow[r, "\tilde{\lambda}"] & \wedge^n T^*BP
\end{tikzcd}$$

The condition $\wedge^n T^*Bf \circ \tilde{\lambda} = \tilde{\lambda} \circ J^rCf$ for all automorphisms $f \in \mathrm{Aut}(P)$ is equivalent to a Lagrangian λ of order r on C being gauge natural. Thus we obtain that a Lagrangian λ of order r on C is gauge natural iff $(J^rCf)^*\lambda = \lambda$ holds for all automorphisms $f \in \mathrm{Aut}(P)$.

Secondly we prove that $J^3g^*\Theta_\lambda = \Theta_{J^2g^*\lambda}$ for all local automorphisms $g \in \mathrm{Aut}(C)$. Let (V,ψ), $\psi = (x^i, y^\sigma)$ and $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$ be two fibered charts such that $g(V) \subset \bar{V}$. We write $\bar{x}^i \circ g_0 \circ \phi^{-1} = \bar{x}^i$ where $\phi = (x^i)$ and $\bar{y}^\sigma \circ g \circ \psi^{-1} = \bar{y}^\sigma$. Now we can use the transformation rules for the jet coordinates and the contact forms from Theorem 3.1 and we obtain in the obvious shorthand

[1] In what follows we write C for the corresponding functor too.

notation

$$\Theta_{J^2g^*\lambda} = \bar{\mathcal{L}}J\omega_0 + \left(\frac{\partial\bar{\mathcal{L}}J}{\partial y_i^\sigma} - d_p\frac{\partial\bar{\mathcal{L}}J}{\partial y_{pi}^\sigma}\right)\eta^\sigma\wedge\omega_i + \frac{\partial\bar{\mathcal{L}}J}{\partial y_{ji}^\sigma}\eta_j^\sigma\wedge\omega_i$$

$$= \bar{\mathcal{L}}J\omega_0 + \left(\frac{\partial\bar{\mathcal{L}}}{\partial y_i^\sigma}J - d_p\frac{\partial\bar{\mathcal{L}}J}{\partial y_{pi}^\sigma}\right)\eta^\sigma\wedge\omega_i + \frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{qm}^\rho}\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^j}{\partial\bar{x}^q}\frac{\partial x^i}{\partial\bar{x}^m}J\eta_j^\sigma\wedge\omega_i$$

$$J^3g^*\Theta_\lambda = J^3g^*\left(\bar{\mathcal{L}}\bar{\omega}_0 + \left(\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_i^\sigma} - \bar{d}_p\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{pi}^\sigma}\right)\bar{\eta}^\sigma\wedge\bar{\omega}_i + \frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{ji}^\sigma}\bar{\eta}_j^\sigma\wedge\bar{\omega}_i\right)$$

$$= \bar{\mathcal{L}}J\omega_0 + \left(\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_i^\sigma} - \bar{d}_p\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{pi}^\sigma}\right)\frac{\partial\bar{y}^\sigma}{\partial y^\nu}\eta^\nu\wedge\frac{\partial x^j}{\partial\bar{x}^i}J\omega_j + \frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{ji}^\sigma}\left(\frac{\partial\bar{y}^\sigma}{\partial y^\rho}\frac{\partial x^l}{\partial\bar{x}^j}\eta_l^\rho\right.$$

$$\left.+d_l\frac{\partial\bar{y}^\sigma}{\partial y^\rho}\frac{\partial x^l}{\partial\bar{x}^j}\eta^\rho\right)\wedge\frac{\partial x^k}{\partial\bar{x}^i}J\omega_k = \bar{\mathcal{L}}J\omega_0 + \left(\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_i^\sigma} - \bar{d}_p\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{pi}^\sigma}\right)\frac{\partial\bar{y}^\sigma}{\partial y^\nu}\frac{\partial x^j}{\partial\bar{x}^i}J\eta^\nu\wedge\omega_j$$

$$+\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{ji}^\sigma}d_l\frac{\partial\bar{y}^\sigma}{\partial y^\rho}\frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^k}{\partial\bar{x}^i}J\eta^\rho\wedge\omega_k + \frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{ji}^\sigma}\frac{\partial\bar{y}^\sigma}{\partial y^\rho}\frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^k}{\partial\bar{x}^i}J\eta_l^\rho\wedge\omega_k.$$

So the first and the last terms in the expressions for $\Theta_{J^2g^*\lambda}$ and $J^3g^*\Theta_\lambda$ are same. But the following computation shows that the remaining terms are same too:

$$\frac{\partial\bar{\mathcal{L}}}{\partial y_i^\sigma}J - d_p\frac{\partial\bar{\mathcal{L}}J}{\partial y_{pi}^\sigma} = \left(\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_j^\rho}\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^i}{\partial\bar{x}^j} + \frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{jk}^\rho}\left(\frac{\partial^2\bar{y}^\rho}{\partial y^\sigma\partial x^l}\frac{\partial x^i}{\partial\bar{x}^j}\frac{\partial x^l}{\partial\bar{x}^k} + \frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial^2x^i}{\partial\bar{x}^j\partial\bar{x}^k}\right.\right.$$

$$\left.\left.+\frac{\partial^2\bar{y}^\rho}{\partial x^l\partial y^\sigma}\frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k} + \frac{\partial^2\bar{y}^\rho}{\partial y^\sigma\partial y^\nu}y_l^\nu\left(\frac{\partial x^i}{\partial\bar{x}^j}\frac{\partial x^l}{\partial\bar{x}^k} + \frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k}\right)\right)\right)J$$

$$-d_p\left(\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{jk}^\rho}\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^p}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k}\right)J - \frac{\partial\bar{\mathcal{L}}}{\partial y_{pi}^\sigma}d_pJ = \left(\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_j^\rho} - \bar{d}_q\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{qj}^\rho}\right)\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^i}{\partial\bar{x}^j}J$$

$$+\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{jk}^\rho}\left(\frac{\partial^2\bar{y}^\rho}{\partial y^\sigma\partial x^l}\frac{\partial x^i}{\partial\bar{x}^j}\frac{\partial x^l}{\partial\bar{x}^k} + \frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial^2x^i}{\partial\bar{x}^j\partial\bar{x}^k} + \frac{\partial^2\bar{y}^\rho}{\partial x^l\partial y^\sigma}\frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k}\right.$$

$$+2\frac{\partial^2\bar{y}^\rho}{\partial y^\sigma\partial y^\nu}y_l^\nu\frac{\partial x^i}{\partial\bar{x}^j}\frac{\partial x^l}{\partial\bar{x}^k} - d_p\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^p}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k} - \frac{\partial\bar{y}^\rho}{\partial y^\sigma}d_p\frac{\partial x^p}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k}$$

$$\left.-\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^p}{\partial\bar{x}^j}d_p\frac{\partial x^i}{\partial\bar{x}^k} - \frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^p}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k}\frac{\partial x^q}{\partial\bar{x}^l}\frac{\partial^2\bar{x}^l}{\partial x^p\partial x^q}\right)J$$

$$= \left(\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_j^\rho} - \bar{d}_q\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{qj}^\rho}\right)\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^i}{\partial\bar{x}^j}J + \frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{jk}^\rho}d_p\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\frac{\partial x^p}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k}J$$

$$+\frac{\partial\bar{\mathcal{L}}}{\partial\bar{y}_{jk}^\rho}\frac{\partial\bar{y}^\rho}{\partial y^\sigma}\left(\frac{\partial^2x^i}{\partial\bar{x}^j\partial\bar{x}^k} - d_p\frac{\partial x^p}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k} - \frac{\partial x^p}{\partial\bar{x}^j}d_p\frac{\partial x^i}{\partial\bar{x}^k} - \frac{\partial x^p}{\partial\bar{x}^j}\frac{\partial x^i}{\partial\bar{x}^k}\frac{\partial x^q}{\partial\bar{x}^l}\frac{\partial^2\bar{x}^l}{\partial x^p\partial x^q}\right)J,$$

where the last term is zero because

$$\frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k} - d_p \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^i}{\partial \bar{x}^k} - \frac{\partial x^p}{\partial \bar{x}^j} d_p \frac{\partial x^i}{\partial \bar{x}^k} - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^q}{\partial \bar{x}^l} \frac{\partial^2 \bar{x}^l}{\partial x^p \partial x^q}$$
$$= -\frac{\partial x^i}{\partial \bar{x}^k} \left(\frac{\partial^2 \bar{x}^l}{\partial \bar{x}^q \partial \bar{x}^j} \frac{\partial \bar{x}^q}{\partial x^l} + \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^l} \frac{\partial^2 \bar{x}^l}{\partial x^p \partial x^q} \right) = -\frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial}{\partial \bar{x}^j} \left(\frac{\partial x^l}{\partial \bar{x}^q} \frac{\partial \bar{x}^q}{\partial x^l} \right) = 0.$$

Thus we really have $J^3 g^* \Theta_\lambda = \Theta_{J^2 g^* \lambda}$ for all local automorphisms $g \in \mathrm{Aut}(C)$.

If a Lagrangian λ of order 2 on C is gauge natural, then combining both statements together we get $J^3 Cf^* \Theta_\lambda = \Theta_{(J^2 Cf)^* \lambda} = \Theta_\lambda$ for all local automorphisms $f \in \mathrm{Aut}(P)$, i.e. λ is $\mathrm{Aut}(P)$-invariant. Conversely, if λ is $\mathrm{Aut}(P)$-invariant, then we have $\Theta_{(J^2 Cf)^* \lambda} = J^3 Cf^* \Theta_\lambda = \Theta_\lambda$ for all local automorphisms $f \in \mathrm{Aut}(P)$. If we apply the horizontalization, then we see that $(J^2 Cf)^* \lambda = \lambda$ holds for all local automorphisms $f \in \mathrm{Aut}(P)$. Therefore the first statement implies that λ is gauge natural and it finishes the proof.

$\diamondsuit$

The definition of the gauge natural structure given here differs from the definition used in [16]. In [16], the authors assign to Lagrangian λ of order r a form Θ_λ and define λ to be $\mathrm{Aut}(P)$**-covariant**, if $(J^{r-1} Cf)^* \Theta_\lambda = \Theta_\lambda$ for all $f \in \mathrm{Aut}(P)$; instead of condition 3 they apply the $\mathrm{Aut}(P)$-covariance condition. They also apply one more condition which we do not need here.

In [16] the authors introduce the Lepage form (3.5) and higher order Lepage forms without knowledge of original sources [28, 29]. However the generalization of Θ_λ as well as the covariance condition in higher order gauge natural field theories (Lagrangian symmetries) are unclear.

3.3 Noether's Theorem and Induced Variations

If we want to write Noether's theorem, we have to introduce the notion of an invariance transformation. If Y denotes a differentiable manifold and f a local diffeomorphism of Y and $f^* \rho = \rho$ holds for a differential form ρ on Y, then f is called an **invariance transformation** of the differential form ρ. In the calculus of variations we deal with fibered manifolds and we use their local automorphisms, which transform cross sections into cross sections. If $J^r f^* \rho = \rho$ holds for a (local) automorphism $f \in \mathrm{Aut}(Y)$ of the fibered manifold Y and $\rho \in \Omega^r_p W$, then we just say that f (instead of $J^r f$) is an invariance transformation of ρ. Let ξ be a projectable vector field on Y. We say that ξ is the **generator of invariance transformations** of ρ, if $\partial_{J^r \xi} \rho = 0$. This notion includes the invariance of a Lagrangian or the **Euler-Lagrange form** E_λ given by the relation $E_\lambda = p_1 d\Theta_\lambda$ for a Lepage equivalent of a Lagrangian λ. We say that a section $\gamma \in \Gamma_{\Omega, W}(Y)$ is an **extremal** of the variational function λ_Ω corresponding to a Lagrangian $\lambda \in \Omega^r_{n,X} W$, if it is a stable point with respect to all its variations induced by a vector field ξ with support contained in $\pi^{-1}(\Omega)$. A section γ is called simply an extremal, if it is an extremal for every variational function λ_Ω. It is a consequence of the integral first variation formula (see [30] or Equation (3.13) below) that γ is an extremal of λ_Ω if and only if one of the following conditions holds: [2]

1. For every π-vertical vector field ξ satisfying the condition on ξ in the definition of an extremal of λ_Ω we have $J^{r-1}\gamma^* i_{J^{r-1}\xi} d\Theta_\lambda = 0$.

2. For any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, γ satisfies the system of partial differential equations

$$E_\sigma(\mathcal{L}) \circ J^r\gamma = 0, \ 1 \leq \sigma \leq m, \tag{3.9}$$

where $E_\sigma(\mathcal{L})$ are the so called **Euler-Lagrange expressions**, which appear in the relation (also proved in [30])

$$p_1 d\Theta_\lambda = E_\sigma(\mathcal{L})\eta^\sigma \wedge \omega_0. \tag{3.10}$$

For the principal Lepage equivalent from Theorem 3.1 we have $E_\sigma(\mathcal{L}) = \left(\frac{\partial \mathcal{L}}{\partial y^\sigma} - d_i\left(\frac{\partial \mathcal{L}}{\partial y^\sigma_i} - d_p \frac{\partial \mathcal{L}}{\partial y^\sigma_{pi}}\right)\right)$ (compare with Theorem 3.2).

3. The Euler-Lagrange form associated with λ vanishes along $J^r\gamma$, i.e.

$$E_\lambda \circ J^r\gamma = 0. \tag{3.11}$$

It can be proved (see [28]) that if g is an invariance transformation of a Lagrangian λ, then it is an invariance transformation of its Euler-Lagrange form E_λ. Moreover, if g is an invariance transformation of the Euler-Lagrange form E_λ, then it brings an extremal γ into an extremal, i.e. $g \circ \gamma \circ g_0^{-1}$ is an extremal too.

Theorem 3.4. *(Noether's theorem) Let $\Theta_\lambda \in \Omega_n^{r-1}W$ be a Lepage equivalent of a Lagrangian $\lambda \in \Omega^r_{n,X}W$ and let γ be an extremal. For any generator ξ of invariance transformations of λ*

$$dJ^{r-1}\gamma^* i_{J^{r-1}\xi}\Theta_\lambda = 0 \tag{3.12}$$

holds.

Proof: It is a consequence of the integral first variation formula

$$(\partial_{J^r\xi}\lambda)_\Omega(\gamma) = \int_\Omega J^{r-1}\gamma^* i_{J^{r-1}\xi} d\Theta_\lambda + \int_\Omega J^{r-1}\gamma^* d i_{J^{r-1}\xi}\Theta_\lambda.$$

The left hand side vanishes because ξ is a generator of invariance transformations of λ, i.e. $\partial_{J^r\xi}\lambda = 0$. The first term on the right hand side vanishes because γ is

[2] We suppose that the order of the Lepage equivalent is $r-1$ as before in the first variation formula (3.8). But generally, if the Lagrangian is of order r, then the Euler-Lagrange equations are of order less or equal to $2r$. So in Equations (3.9) and (3.11) we should write $J^{2r}\gamma$ instead of $J^r\gamma$.

an extremal. In fact, we have

$$\begin{aligned}\int_\Omega J^{r-1}\gamma^* i_{J^{r-1}\xi} d\Theta_\lambda &= \int_\Omega J^r\gamma^*\pi^{r,r-1*} i_{J^{r-1}\xi} d\Theta_\lambda = \int_\Omega J^r\gamma^* i_{J^r\xi}\pi^{r,r-1*} d\Theta_\lambda \\ &= \int_\Omega J^r\gamma^* i_{J^r\xi} p_1 d\Theta_\lambda = \int_\Omega J^r\gamma^* i_{J^r\xi}(E_\sigma(\mathcal{L})\eta^\sigma \wedge \omega_0) \\ &= \int_\Omega E_\sigma(\mathcal{L})\circ J^r\gamma\cdot\left(\xi^\sigma\circ\gamma - \frac{\partial(y^\sigma\circ\gamma)}{\partial x^j}\xi^j\right)\omega_0 = 0, \end{aligned} \tag{3.13}$$

where we have used the canonical decomposition of $\pi^{r,r-1*}d\Theta_\lambda$ into the horizontal and contact component. The horizontal part is zero, since it is the horizontalization of an $n+1$ form and from the contact component only the 1-contact part survives the pullback by $J^r\gamma$. We have also made use of Equation (3.10) and (3.9). Thus the second term on the right hand side of the integral first variation formula must be zero. Therefore we see that $J^{r-1}\gamma^* d i_{J^{r-1}\xi}\Theta_\lambda = dJ^{r-1}\gamma^* i_{J^{r-1}\xi}\Theta_\lambda = 0$ holds for any generator ξ of invariance transformations of L.

$\diamondsuit$

Equation (3.12) from Noether's theorem is called **(differential) conservation law**. The term $i_{J^{r-1}\xi}\Theta_\lambda$ in the conservation law is called the **current**. The case when the assumptions are relaxed to only ξ being a generator of invariance transformations of E_λ in Theorem 3.4 (Noether-Bessel-Hagen theorem) is treated in [28], [8] or [25].

Now we will discuss the first variation formula for the so called induced variations for a gauge natural Lagrangian λ. By the **induced variation** we mean the variation induced by the lifted [3] vector field $C\xi$ on the configuration bundle C determined by an **(infinitesimal) generator of automorphisms** ξ on a principal bundle P which is a vector field such that $\mathrm{Fl}^\xi_t \in \mathrm{Aut}(P)$. We have seen in the first part of the proof of Theorem 3.3 that a Lagrangian λ of order r on C is gauge natural iff $(J^rCf)^*\lambda = \lambda$ holds for all local automorphisms $f \in \mathrm{Aut}(P)$, so Cf is an invariance transformation of the Lagrangian λ. If ξ is a generator of automorphisms on P, then we have $(\mathrm{Fl}^{J^rC\xi}_t)^*\lambda = (J^rC\mathrm{Fl}^\xi_t)^*\lambda = \lambda$, so differentiating at $t=0$ we obtain

$$\partial_{J^rC\xi}\lambda = 0, \tag{3.14}$$

thus $C\xi$ is the generator of invariance transformations of λ and Equation (3.14) is called the **covariance identity**.

Now we will find for the next computations a local expression of the generator of automorphisms ξ on a principal bundle P. First we show that locally we can write

$$\phi_\alpha \circ \mathrm{Fl}^\xi_t \circ \phi_\alpha^{-1}(x,a) = ((\mathrm{Fl}^\xi_t)_0(x), (\mathrm{Fl}^\xi_t)_1(x)a) \tag{3.15}$$

[3]Here lifting means acting by the flow operator corresponding to a configuration bundle C.

for all $(x,a) \in U_\alpha \times G$, where $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a local trivialization of P. Certainly there exists a map $(\mathrm{Fl}^\xi_t)'_1 : U_\alpha \times G \to G$ such that

$$\phi_\alpha \circ \mathrm{Fl}^\xi_t \circ \phi_\alpha^{-1}(x,a) = ((\mathrm{Fl}^\xi_t)_0(x), a(\mathrm{Fl}^\xi_t)'_1(x,a)) \tag{3.16}$$

for all $(x,a) \in U_\alpha \times G$. The computation ($\cdot$ is the principal right action on P)

$$\begin{aligned}((\mathrm{Fl}^\xi_t)_0(x), ab(\mathrm{Fl}^\xi_t)'_1(x,ab)) &= \phi_\alpha \circ \mathrm{Fl}^\xi_t \circ \phi_\alpha^{-1}(x,ab) = \phi_\alpha \circ \mathrm{Fl}^\xi_t(\phi_\alpha^{-1}(x,a)\cdot b)\\ &= \phi_\alpha(\mathrm{Fl}^\xi_t(\phi_\alpha^{-1}(x,a))\cdot b) = \phi_\alpha(\phi_\alpha^{-1}((\mathrm{Fl}^\xi_t)_0(x), a(\mathrm{Fl}^\xi_t)'_1(x,a))\cdot b)\\ &= ((\mathrm{Fl}^\xi_t)_0(x), a(\mathrm{Fl}^\xi_t)'_1(x,a)b)\end{aligned}$$

shows that $(\mathrm{Fl}^\xi_t)'_1(x,ab) = b^{-1}(\mathrm{Fl}^\xi_t)'_1(x,a)b$ for all $x \in U_\alpha$, $a,b \in G$. This could be seen at once from the properties of $\tau : (U_\alpha \times G) \times_{U_\alpha} (U_\alpha \times G) \to G$ from page 11, if we realize that $(\mathrm{Fl}^\xi_t)'_1(x,a) = \tau(((\mathrm{Fl}^\xi_t)_0(x),a), \phi_\alpha \circ \mathrm{Fl}^\xi_t \circ \phi_\alpha^{-1}(x,a))$. If we set $(\mathrm{Fl}^\xi_t)_1(x) = (\mathrm{Fl}^\xi_t)'_1(x,e)$, then we have $(\mathrm{Fl}^\xi_t)'_1(x,a) = a^{-1}(\mathrm{Fl}^\xi_t)_1(x)a$ and so we obtain from relation (3.16) Equation (3.15). Now we write

$$\xi_0(x) = \left(\frac{d}{dt}(\mathrm{Fl}^\xi_t)_0(x)\right)_0 = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{3.17}$$

$$\xi_1(x) = \left(\frac{d}{dt}(\mathrm{Fl}^\xi_t)_1(x)\right)_0 = \xi^P(x)e_P, \tag{3.18}$$

where e_P, $1 \le P \le \dim G$ is a basis of $\mathfrak{g}$. Furthermore, ρ is the right translation, $R_P(a) = R_{e_P}(a) = T_e\rho_a(e_P)$ denotes the right invariant vector field on G corresponding to e_P. So we get, differentiating Equation (3.15) at $t = 0$, the local expression of the generator of automorphisms ξ

$$\begin{aligned}\xi(x,a) = \xi_0(x) + T_e\rho_a(\xi_1(x)) &= \xi^i(x)\frac{\partial}{\partial x^i} + \xi^P(x)T_e\rho_a(e_P)\\ &= \xi^i(x)\frac{\partial}{\partial x^i} + \xi^P(x)R_P(a).\end{aligned} \tag{3.19}$$

Chapter 4

The Hilbert-Yang-Mills Functional

We analyze the gauge natural structure of the Einstein-Yang-Mills theory, which describes the interaction of gravity with the Yang-Mills field. We introduce the Hilbert-Yang-Mills functional, whose Lagrangian λ is given by the sum of the Hilbert Lagrangian and the Yang-Mills Lagrangian. We derive the principal Lepage equivalent of the Hilbert-Yang-Mills Lagrangian and the corresponding first variation formula. We study the invariance of λ with respect to automorphisms of a structure bundle and we discuss the first variation formula for induced variations. We show that the currents in the Einstein-Yang-Mills theory split into three summands, one of which is the exterior derivative of the Komar-Yang-Mills superpotential.

4.1 The Gauge Natural Structure of Einstein-Yang-Mills Theory

First we describe the gauge natural structure of Einstein-Yang-Mills theory. Let (P, p, X, G) be a structure bundle, where the n-dimensional manifold X is interpreted as spacetime. We shall consider Yang-Mills theories with a general Lie group G. The configuration bundle for the Yang-Mills part is $C_e = QP$, where QP is the bundle of principal connections. The configuration bundle for the gravitational part is $C_g = F^1X \times_{l_1} \mathrm{LMet}\,\mathbb{R}^n$, where $\mathrm{LMet}\,\mathbb{R}^n$ is the set of bilinear, symmetric, non-degenerate forms with the Lorentzian signature

$(1, n-1)$ and $l_1 : L^1_n \times \mathrm{LMet}\,\mathbb{R}^n \to \mathrm{LMet}\,\mathbb{R}^n$, $l_1(a,g) = g \circ (a^{-1} \times a^{-1})$ is a left action ($L^1_n \cong GL(\mathbb{R}^n)$). We shall take $C = C_g \times_X C_e$ as the configuration bundle for gravitation and Yang-Mills theory, which we call **Einstein-Yang-Mills theory**. With respect to Theorem 2.8 and Theorem 2.7 we see that C is the gauge natural bundle of order 1 and $C \cong W^1P \times_l (\mathrm{LMet}\,\mathbb{R}^n \times S)$.

We wish to discuss the condition 3 in the definition of the gauge natural structure in more detail. By describing all possible gauge natural Lagrangians we will immediately see that the Hilbert-Yang-Mills Lagrangian has the required form. We assume that the interaction Lagrangian for the Einstein-Yang-Mills theory is of first order and we look for all first order gauge natural operators $C \to \wedge^n T^*B$ from the configuration bundle C of the Einstein-Yang-Mills theory. In what follows we prove the Utiyama-like theorem. Utiyama's theorem [57] was reproved and generalized by many authors. Our proof is based on the Utiyama-like theorem in [25] for the bundle of principal connections implemented with the gravitational part. For a higher order version of the Utiyama-like theorem see [24]. Our assertion agrees with the result given in [16] where the authors used a different method.

We will apply the **orbit reduction** , i.e. the following theorem [25, 34]. Let $G, \tilde{G}$ be Lie groups and let $p : G \to \tilde{G}$ be a surjective Lie group homomorphism with kernel K. Let M be a left G-manifold and $Q, \tilde{M}$ left $\tilde{G}$-manifolds. Then we can define a left action of the group G on the manifold $\tilde{M}$ by $g \cdot y = p(g) \cdot y$ for $g \in G$, $y \in \tilde{M}$. Thus, $\tilde{M}$ becomes a left G-manifold. Let $\pi : M \to Q$ be a p-equivariant surjective submersion, i.e. $\pi(g \cdot x) = p(g) \cdot \pi(x)$ for all $g \in G$, $x \in M$.

Theorem 4.1. *(orbit reduction) If for every point $q \in Q$ the set $\pi^{-1}(q)$ is a K-orbit in M, then there exists a bijection between smooth G-equivariant maps $f : M \to \tilde{M}$ and smooth $\tilde{G}$-equivariant maps $f' : Q \to \tilde{M}$ given by $f = f' \circ \pi$, i.e. we have the commutative factorization diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & \tilde{M} \\ Q \arrow[ur, "f'"'] & \end{tikzcd}$$

Proof: For every smooth $\tilde{G}$-equivariant map $f' : Q \to \tilde{M}$ we have $f' \circ \pi(g \cdot x) = f'(p(g) \cdot \pi(x)) = p(g) \cdot (f' \circ \pi(x))$, so $f' \circ \pi$ is a smooth G-equivariant map. Conversely, for every G-equivariant map $f : M \to \tilde{M}$ we can define $f' : Q \to \tilde{M}$ by $f'(\pi(x)) = f(x)$. Such an f' is well defined, in fact, for another representative $\pi(\tilde{x}) = \pi(x)$, i.e. $\tilde{x}, x \in \pi^{-1}(q) = \mathrm{orb}_K(x)$ we have $\tilde{x} = kx$ with $k \in K$, thus we get $f'(\pi(\tilde{x})) = f'(\pi(k \cdot x)) = f(k \cdot x) = p(k) \cdot f(x) = e \cdot f(x) = f(x)$. Since π is a surjective submersion and $f = f' \circ \pi$ is smooth, f' is smooth by the universal property of surjective submersion. Since p and π are surjective, there exists for all $\tilde{g} \in \tilde{G}$ and $q \in Q$ some $g \in G$ and $x \in M$ such that $p(g) = \tilde{g}$ and $\pi(x) = q$, hence we have $f'(\tilde{g} \cdot q) = f'(p(g) \cdot \pi(x)) = f'(\pi(g \cdot x)) = f(g \cdot x) = p(g) \cdot f(x) =$

$\tilde{g} \cdot f'(q)$, thus f' is $\tilde{G}$-equivariant. The uniqueness of the f' follows from the surjectivity of π, therefore we have proved that $f \mapsto f'$ is a bijection.

$\diamondsuit$

From Theorem 2.9 and Theorem 2.11 we see that first order gauge natural operators $C \to \wedge^n T^*B$ are in canonical bijection with $W^2_n G$-equivariant maps $J^1_0 C \to \wedge^n T^* B_0$ between the standard fibers $J^1_0 C = J^1_0 C(\mathbb{R}^n \times G)$, $\wedge^n T^* B_0 = \wedge^n T^* B_0(\mathbb{R}^n \times G)$. Using Theorem 2.4 we get the left $W^2_n G$-manifold $J^1_0 C \cong T^1_n(\mathrm{LMet}\mathbb{R}^n \times S)$ with the left action l^1 given by Equation (2.6), further we have the left L^1_n-manifold $\wedge^n T^* B_0 \cong \mathbb{R}$ with the action given by

$$\bar{t} = |\det a|^{-1} t \tag{4.1}$$

where $t, \bar{t} \in \mathbb{R}$ and $a \in L^1_n$.

Now we are going to describe the action l_2 from Theorem 2.7 in more detail. For the fiber we get $S \cong J^1_0(\mathbb{R}^n, G)_e \cong L(T_0\mathbb{R}^n, T_e G) \cong \mathrm{L}(\mathbb{R}^n, \mathfrak{g}) \cong \mathfrak{g} \otimes \mathbb{R}^{n*}$ and for the group $W^1_n G \cong L^1_n \rtimes T^1_n G \cong (L^1_n \times G) \rtimes (\mathfrak{g} \otimes \mathbb{R}^{n*})$, where in the last identification we use the isomorphism $T^r_n G \to G \rtimes J^r_0(\mathbb{R}^n, G)_e$ given by $J^r_0 s \mapsto (s(0), J^r_0(s(0)^{-1} \cdot s))$. We want to express the action $l_2((A, a, Z), Y)$ with $(A, a, Z) \cong J^1_{(0,e)}\phi \in W^1_n G$, $A = J^1_0 \phi_0$, $a = \mathrm{pr}_2 \circ \phi(0, e)$, $Z = T_0(a^{-1}\phi_1)$ and $Y = T_0 \bar{s} \in S$, where $\bar{s} = \mathrm{pr}_2 \circ s$ for $s : \mathbb{R}^n \to \mathbb{R}^n \times G$ such that $s(0) = (0, e)$. By definition $Q\phi(J^1_0 s) = [J^1_0(\rho_{a^{-1}} \circ \phi \circ s \circ \phi_0^{-1})]$, where ρ denotes the principal right action of G. Then we evaluate $\mathrm{pr}_2 \circ \rho_{a^{-1}} \circ \phi \circ s(x) = \mathrm{pr}_2 \circ \rho_{a^{-1}} \circ \phi \circ \rho_{\bar{s}(x)}(x, e) = \mathrm{pr}_2 \circ \rho_{a^{-1}} \circ \rho_{\bar{s}(x)}(\phi_0(x), \phi_1(x)) = \phi_1(x)\bar{s}(x)a^{-1} = \mathrm{conj}_a \circ \mu \circ (a^{-1}\phi_1, \bar{s})(x)$, with μ denoting the multiplication in the group G and conj_a is the **conjugation** (the **inner automorphism**) associated with $a \in G$ defined by $\mathrm{conj}_a : G \to G;\ \mathrm{conj}_a(x) = axa^{-1}$. By applying the tangent functor we find that

$$\begin{aligned} l_2((A, a, Z), Y) &= T_0(\mathrm{pr}_2 \circ \rho_{a^{-1}} \circ \phi \circ s \circ \phi_0^{-1}) = T_0(\mathrm{conj}_a \circ \mu \circ (a^{-1}\phi_1, \bar{s}) \circ \phi_0^{-1}) \\ &= T_e \mathrm{conj}_a \circ T_{(e,e)}\mu(T_0(a \cdot \phi_1), T_0\bar{s}) \circ T_0\phi_0^{-1}) = \mathrm{Ad}(a)(Y + Z) \circ A^{-1}, \end{aligned}$$

where $\mathrm{Ad} : G \to GL(\mathfrak{g});\ \mathrm{Ad}(a) = T_e \mathrm{conj}_a$ is the **adjoint representation** of G and in the last equality we use the relation $T_{(a,b)}\mu(X_a, Y_b) = T_a(\rho_b)X_a + T_b(\lambda_a Y_b)$, here $\lambda_a : G \to G,\ \lambda_a(x) = ax$ denotes left translation and $\rho_a : G \to G,\ \rho_a(x) = xa$ right translation.

Denote by e_i the canonical basis of $\mathbb{R}^n$ and e^i the dual basis of $\mathbb{R}^n$ and e_P the basis of $\mathfrak{g}$. Each element $g \in \mathrm{LMet}\mathbb{R}^n$ is then uniquely written in the form $g = g_{ij}(g) e^i \odot e^j$ and each element $\Gamma \in \mathrm{L}(\mathbb{R}^n, \mathfrak{g})$ is uniquely written in the form $\Gamma(e_i) = \Gamma^P_i(\Gamma) e_P$, where $1 \le i \le j \le n$ and $1 \le P \le m = \dim G$. The system of functions (g_{ij}, Γ^P_i) defines a global chart on $\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g})$. So there exists a canonical global chart

$$\begin{aligned} &g_{ij}(J^1_0 s) = g_{ij}(s(0)),\ g_{ij,k}(J^1_0 s) = D_k(g_{ij} s)(0), \\ &\Gamma^P_i(J^1_0 s) = \Gamma^P_i(s(0)),\ \Gamma^P_{ij}(J^1_0 s) = D_j(\Gamma^P_i s)(0) \end{aligned}$$

on $T^1_n(\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g}))$. The coordinate functions on $W^2_n G$ are defined by

$$a^i_j(J^2_{(0,e)}\phi) = D_j\phi^i_0(0),\ a^P_i(J^2_{(0,e)}\phi) = D_i(a^{-1}\phi_1)^P(0),$$
$$a^i_{jk}(J^2_{(0,e)}\phi) = D_jD_k\phi^i_0(0),\ a^P_{ij}(J^2_{(0,e)}\phi) = D_iD_j(a^{-1}\phi_1)^P(0),$$

where $a = \mathrm{pr}_2 \circ\phi(0, e)$. The action of $W^2_n G$ on $T^1_n(\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g}))$ is given by Equation (2.6), i.e.

$$l^1(J^2_{(0,e)}\phi, J^1_0 s) = J^1_0(l \circ ((b^{-1} \circ W^1\phi \circ b)_1 \circ \phi_0^{-1}, s \circ \phi_0^{-1})), \tag{4.2}$$

where b denotes the identification $b : \mathbb{R}^n \times W^r_n G \ni (x, J^r_{(0,e)}\phi) \mapsto J^r_{(0,e)}(\tau_x \circ \phi) \in W^r(\mathbb{R}^n\times G)$ for $r = 1$, $\tau_x = t_x\times\mathrm{id}_G$ and t_x is the translation $\mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto y+x$, so we use the inclusion $W^2_n G \hookrightarrow W^1_n W^1_n G$, $J^2_{(0,e)}\phi \mapsto J^1_{(0,e)}(b^{-1} \circ W^1\phi \circ b)$, and l is induced by l_1 and l_2 (see the proof of Theorem 2.8). Now we want to express Equation (4.2) in coordinates. For some $J^1_{(0,e)}\phi \in W^1_n G$ and $(g, \Gamma) \in \mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g})$ our l_1 and l_2 have in coordinates the form

$$g_{ij}(J^1_{(0,e)}\phi \cdot g) = \tilde{a}^k_i(J^1_{(0,e)}\phi)\tilde{a}^l_j(J^1_{(0,e)}\phi)g_{kl}(g), \tag{4.3}$$
$$\Gamma^P_i(J^1_{(0,e)}\phi \cdot \Gamma) = \mathrm{A}^P_Q(\mathrm{pr}_2 \circ\beta(J^1_{(0,e)}\phi))(\Gamma^Q_j(\Gamma) + a^Q_j(J^1_{(0,e)}\phi))\tilde{a}^j_i(J^1_{(0,e)}\phi), \tag{4.4}$$

where $\tilde{a}^j_i(J^1_{(0,e)}\phi) = D_i\phi_0^{-1j}(0)$ and A^P_Q is the coordinate expression of the adjoint representation of G. Using the identity $(b^{-1} \circ W^1\phi \circ b)_1 \circ \phi_0^{-1}(x) = J^1_{(0,e)}(\tau_x^{-1} \circ \phi \circ \tau_{\phi_0(x)})$ we deduce that the action of $W^2_n G$ on $T^1_n(\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g}))$ has the form

$$\begin{aligned}
\bar{g}_{ij} &= \tilde{a}^k_i\tilde{a}^l_j g_{kl},\\
\bar{g}_{ij,k} &= \tilde{a}^l_i\tilde{a}^m_j\tilde{a}^n_k g_{lm,n} + (\tilde{a}^l_{ki}\tilde{a}^m_j + \tilde{a}^l_i\tilde{a}^m_{kj})g_{lm},\\
\bar{\Gamma}^P_i &= \mathrm{A}^P_Q(a)(\Gamma^Q_j + a^Q_j)\tilde{a}^j_i,\\
\bar{\Gamma}^P_{ij} &= \mathrm{A}^P_Q(a)\Gamma^Q_{kl}\tilde{a}^k_i\tilde{a}^l_j + \mathrm{A}^P_Q(a)a^Q_{kl}\tilde{a}^k_i\tilde{a}^l_j + D^P_{QR}(a)\Gamma^Q_k a^R_l\tilde{a}^k_i\tilde{a}^l_j\\
&\quad + E^P_{QR}(a)a^Q_k a^R_l\tilde{a}^k_i\tilde{a}^l_j + \mathrm{A}^P_Q(a)(\Gamma^Q_k + a^Q_k)\tilde{a}^k_{ij},
\end{aligned}$$

where we introduced shorthand notation, e.g. $g_{ij} = g_{ij}(J^1_0 s)$, $\bar{g}_{ij} = g_{ij}(J^2_{(0,e)}\phi \cdot J^1_0 s)$, $a^i_j = a^i_j(J^2_{(0,e)}\phi)$, $\Gamma^P_i = \Gamma^P_i(J^1_0 s)$ etc., $D^P_{QR}(a) = D_R\mathrm{A}^P_Q(a)$, $E^P_{QR}(a) = D_R\mathrm{A}^P_Q(a) + \mathrm{A}^P_S(a)D_R(D_Q((a\cdot(.))^{-1}\cdot a\cdot(.))^S(e))(e)$, the first input corresponds to D_R and the second input corresponds to D_Q.

Before we apply orbit reduction we replace the coordinates $(g_{ij}, g_{ij,k}, \Gamma^P_i, \Gamma^P_{ij})$ on $T^1_n(\mathrm{LMet}\mathbb{R}^n\times\mathrm{L}(\mathbb{R}^n, \mathfrak{g}))$ by $(g_{ij}, \Gamma_{i,jk} = \frac{1}{2}(g_{ij,k}+g_{ik,j}-g_{jk,i}), \Gamma^P_i, R^P_{ij} = \Gamma^P_{[ij]}+ c^P_{QR}\Gamma^Q_i\Gamma^R_j, S^P_{ij} = \Gamma^P_{(ij)})$, where $[\ ]$ denotes antisymetrisation, $(\ )$ symetrisation (both without a factor 1/2) and c^P_{QR} the structure constants of G. Then the action of W_n^2G on $T_n^1(\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g}))$ has the form

$$\begin{aligned}
\bar{g}_{ij} &= \tilde{a}_i^k \tilde{a}_j^l g_{kl}, \\
\bar{\Gamma}_{i,jk} &= \tilde{a}_i^l \tilde{a}_j^m \tilde{a}_k^n \Gamma_{l,mn} + \tilde{a}_i^l \tilde{a}_{kj}^m g_{lm}, \\
\bar{\Gamma}_i^P &= \mathrm{A}_Q^P(a)(\Gamma_j^Q + a_j^Q)\tilde{a}_i^j, \\
\bar{R}_{ij}^P &= \mathrm{A}_Q^P(a) R_{kl}^Q \tilde{a}_i^k \tilde{a}_j^l, \\
\bar{S}_{ij}^P &= \mathrm{A}_Q^P(a) S_{kl}^Q \tilde{a}_i^k \tilde{a}_j^l + \mathrm{A}_Q^P(a) a_{kl}^Q \tilde{a}_{(i}^k \tilde{a}_{j)}^l + D_{QR}^P(a)\Gamma_k^Q a_l^R \tilde{a}_{(i}^k \tilde{a}_{j)}^l \\
&\quad + E_{QR}^P(a) a_k^Q a_l^R \tilde{a}_{(i}^k \tilde{a}_{j)}^l + \mathrm{A}_Q^P(a)(\Gamma_k^Q + a_k^Q)\tilde{a}_{ij}^k.
\end{aligned}$$

We define $\pi : T_n^1(\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g})) \to \mathrm{LMet}\mathbb{R}^n \times (\mathfrak{g} \otimes \wedge^2 \mathbb{R}^{n*})$, $(g_{ij}, \Gamma_{i,jk}, \Gamma_i^P, R_{ij}^P, S_{ij}^P) \mapsto (g_{ij}, R_{ij}^P)$, so it is a surjective submersion and $p : W_n^2 G \cong L_n^2 \rtimes T_n^2 G \to L_n^1 \times G$, $p = \pi^{2,1} \times \beta$. In the following theorem $\mathbb{R}$ is considered as the left $L_n^1 \times G$-manifold with the action given by (4.1) (G is acting trivially) and $\mathrm{LMet}\mathbb{R}^n \times (\mathfrak{g} \otimes \wedge^2 \mathbb{R}^{n*})$ is a $L_n^1 \times G$-manifold too (with the action given in the coordinates by $\bar{g}_{ij} = \tilde{a}_i^k \tilde{a}_j^l g_{kl}$ and $\bar{R}_{ij}^P = \mathrm{A}_Q^P(a) R_{kl}^Q \tilde{a}_i^k \tilde{a}_j^l$).

Theorem 4.2. *For every W_n^2G-equivariant map $f : T_n^1(\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g})) \to \mathbb{R}$ there exists a unique $L_n^1 \times G$-equivariant map $f' : \mathrm{LMet}\mathbb{R}^n \times (\mathfrak{g} \otimes \wedge^2 \mathbb{R}^{n*}) \to \mathbb{R}$ satisfying $f = f' \circ \pi$.*

Proof: We apply Theorem 4.1. We see that π is a p-equivariant surjective submersion. Thus we only have to prove that each fiber of π is a K-orbit. The action of K on $T_n^1(\mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n, \mathfrak{g}))$ has the form

$$\begin{aligned}
\bar{g}_{ij} &= g_{ij}, \\
\bar{\Gamma}_{i,jk} &= \Gamma_{i,jk} + \tilde{a}_{kj}^l g_{il}, \\
\bar{\Gamma}_i^P &= \Gamma_i^P + a_i^P, \\
\bar{R}_{ij}^P &= R_{ij}^P, \\
\bar{S}_{ij}^P &= S_{ij}^P + a_{ij}^P + D_{QR}^P(e)\Gamma_{(i}^Q a_{j)}^R + E_{QR}^P(e) a_{(i}^Q a_{j)}^R + (\Gamma_k^P + a_k^P)\tilde{a}_{ij}^k.
\end{aligned}$$

From this we get $\pi^{-1}(g_{ij}, R_{ij}^P) = \mathrm{orb}_K(g_{ij}, 0, 0, R_{ij}^P, 0)$.

$\diamondsuit$

Theorem 4.2 is equivalent to the following modification of Utiyama's assertion.

Theorem 4.3. *For every first order gauge natural Lagrangian of the Einstein-Yang-Mills theory $\lambda : C \to \wedge^n T^*B$ there exists a unique natural transformation $\bar{\lambda} : C_g \times_B ((.) \times_{\mathrm{Ad}} \mathfrak{g} \otimes \wedge^2 T^*B) \to \wedge^n T^*B$ satisfying $\lambda = \bar{\lambda} \circ (\mathrm{id}_{C_g} \times_B R)$, where $R : C_e = Q \to (.) \times_{\mathrm{Ad}} \mathfrak{g} \otimes \wedge^2 T^*B$ is the curvature operator.*

Proof: We apply Theorem 2.9 and Theorem 2.11.

$\diamondsuit$

In other words, our interaction gauge natural Lagrangian depends on its variables only through the metric and the curvature of the principal connection. We can take as the interaction Lagrangian the Yang-Mills Lagrangian i.e. corresponding to the $L^1_n \times G$-equivariant mapping $f' : \mathrm{LMet}\,\mathbb{R}^n \times (\mathfrak{g} \otimes \wedge^2\mathbb{R}^{n*}) \to \mathbb{R}$, $(g_{ij}, R^P_{ij}) \mapsto -(1/4)R^P_{ij}g^{ik}g^{jl}R^Q_{kl}h_{PQ}\sqrt{g}$, where $g = |\det g_{ij}|$ and h denotes an Ad-**invariant form** on the Lie algebra $\mathfrak{g}$, i.e. invariant with respect to the adjoint representation Ad of G in the sense that for all $g \in G$, $X, Y \in \mathfrak{g}$ the relation $h(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)) = h(X, Y)$ holds, and so f' is indeed $L^1_n \times G$-equivariant.

Analogously it can be shown [54] that every L^3_n-equivariant mapping from the left L^3_n-manifold $T^2_n\mathrm{LMet}\,\mathbb{R}^n$ to the left L^1_n-manifold $\mathbb{R}$ depends only on g_{ij} and the curvature R_{ijkl}. We can take as the free Lagrangian the Hilbert Lagrangian corresponding to the L^1_n-equivariant mapping which sends (g_{ij}, R_{ijkl}) to $g^{ij}g^{kl}R_{likj}\sqrt{g}$. Because every gauge natural operator of some order can be considered as an operator of higher order, we can think of the first order interaction Lagrangian as being of second order and we simply add the interaction and free Lagrangian to get the total second order gauge natural Lagrangian.

We now introduce a global variational principle for the Einstein-Yang-Mills equations. Let X be a n-dimensional manifold and let (P, p, X, G) be a structure bundle over X. By the **Hilbert-Yang-Mills Lagrangian** for P we mean the Lagrangian $\lambda = \lambda_H + \lambda_{YM}$ on J^2C, where in any fibered chart (cf. Equation (4.9)) we have

$$\lambda_H = \mathcal{L}_H\omega_0 = R\sqrt{g}\omega_0, \;\; \lambda_{YM} = \mathcal{L}_{YM}\omega_0 = -\frac{1}{4}R^P_{ij}g^{ik}g^{jl}R^Q_{kl}h_{PQ}\sqrt{g}\omega_0. \quad (4.5)$$

In these equations R is the scalar curvature, $R^P_{ij} = \Gamma^P_{j,i} - \Gamma^P_{i,j} + c^P_{QR}\Gamma^Q_i\Gamma^R_j$ is the curvature (field strength) of the principal connection (Yang-Mills field) Γ^P_i, h_{PQ} are the components of an *Ad*-invariant form on the Lie algebra $\mathfrak{g}$. We define the Christoffel symbols by $\Gamma^i_{jk} = (1/2)g^{is}(g_{sj,k} + g_{sk,j} - g_{jk,s})$. The chart expressions of the curvature tensor, the Ricci tensor and the scalar curvature are given by $R^i_{jkl} = d_k\Gamma^i_{jl} - d_l\Gamma^i_{jk} + \Gamma^i_{sk}\Gamma^s_{jl} - \Gamma^i_{sl}\Gamma^s_{jk}$, $R_{jl} = R^m_{jml}$, $R = g^{jl}R_{jl}$.

We summarize that the gauge natural structure of the Einstein-Yang-Mills theory is made of the following items:

1. a structure bundle (P, p, X, G), where the n-dimensional manifold X is interpreted as spacetime,
2. the configuration bundle $C = C_g \times_X C_e \cong W^1P \times_l (\mathrm{LMet}\,\mathbb{R}^n \times S)$ which is a gauge natural bundle of order 1,
3. the Hilbert-Yang-Mills Lagrangian λ on J^2C.

4.2 The Hilbert-Yang-Mills Lagrangian and Induced Variations

We denote the contact forms for the gravitational part by $\eta_{ij} = dg_{ij} - g_{ij,l}dx^l$, $\eta_{ij,k} = dg_{ij,k} - g_{ij,kl}dx^l$ and for the Yang-Mills part by $\eta_i^P = d\Gamma_i^P - \Gamma_{i,j}^P dx^j$ (similarly for the vector field ξ). Furthermore, we denote by $G_{st} = R_{st} - (1/2)g_{st}R$ the components of the Einstein tensor, by $T^{ab} = (1/2)(R^{Pal}R_P{}^b{}_l - (1/4)R_{ij}^P R_P^{ij} g^{ab})$ the components corresponding to the stress-energy tensor and by $\nabla_j R_P^{ji} = d_j R_P^{ji} + R_R^{ji} c_{PS}^R \Gamma_j^S - R_P^{ij}\Gamma_{kj}^k$ the components of the covariant derivative with respect to the C-prolongation of the principal connection with respect to the Levi-Civita connection (see [25]).

Theorem 4.4. *The principal Lepage equivalent Θ_λ of the Hilbert-Yang-Mills Lagrangian has a chart expression*

$$\Theta_\lambda = \sqrt{g}(R - \frac{1}{4}R_{ij}^P R_P^{ij})\omega_0 + \sqrt{g}R_P^{ij}\eta_i^P \wedge \omega_j$$
$$+\frac{1}{2}\sqrt{g}(g^{lm}g^{ia}g^{bj} - 2g^{il}g^{ma}g^{bj} + g^{ma}g^{lb}g^{ij})g_{ml,i}\eta_{ab} \wedge \omega_j$$
$$+\sqrt{g}(g^{ad}g^{jb} - g^{jd}g^{ab})\eta_{ab,d} \wedge \omega_j.$$

The Euler-Lagrange term $hi_\xi d\Theta_\lambda$ has a chart expression

$$hi_\xi d\Theta_\lambda = \sqrt{g}(-G^{ab} + T^{ab})(\xi_{ab} - g_{ab,m}\xi^m)\omega_0 + \sqrt{g}\nabla_j R_P^{ji}(\xi_i^P - \Gamma_{i,m}^P \xi^m)\omega_0.$$

The boundary term $hdi_\xi\Theta_\lambda$ has a chart expression

$$hdi_\xi\Theta_\lambda = d_j\Bigg(\sqrt{g}\left(R - \frac{1}{4}R_{ki}^P R_P^{ki}\right)\xi^j + \sqrt{g}(g^{lb}g^{ij} - g^{il}g^{bj})\Gamma_{li}^a(\xi_{ab} - g_{ab,m}\xi^m)$$
$$+\sqrt{g}R_P^{ij}(\xi_i^P - \Gamma_{i,m}^P\xi^m) + \sqrt{g}(g^{ad}g^{jb} - g^{jd}g^{ba})(\xi_{ab,d} - g_{ab,dm}\xi^m)\Bigg)\omega_0.$$

Proof: We use Equation (3.5) to compute only the Yang-Mills part, because the gravitational part is standard (cf. [35]). We have

$$\frac{\partial \mathcal{L}_{YM}}{\partial \Gamma_i^P} = c_{PS}^R \Gamma_m^S g^{il} g^{mk} R_{kl}^Q h_{RQ}\sqrt{g} = c_{PS}^R \Gamma_m^S R_R^{mi}\sqrt{g}$$
$$\frac{\partial \mathcal{L}_{YM}}{\partial \Gamma_{i,j}^P} = g^{ik}g^{jl}R_{kl}^Q h_{PQ}\sqrt{g} = R_P^{ij}\sqrt{g},$$

thus we get the principal Lepage equivalent corresponding to the Yang-Mills Lagrangian

$$\Theta_{\lambda_{YM}} = -\sqrt{g}\frac{1}{4}R_{ij}^P R_P^{ij}\omega_0 + \sqrt{g}R_P^{ij}\eta_i^P \wedge \omega_j.$$

Now we want to use Theorem 3.2 to compute the Euler-Lagrange term and the boundary term. Using the identities $\partial g^{ik}/\partial g_{ab} = -(1/2)(g^{ia}g^{kb} + g^{ib}g^{ka})$ and $\partial\sqrt{g}/\partial g_{ab} = (1/2)\sqrt{g}g^{ab}$ we get

$$\frac{\partial\mathcal{L}_{YM}}{\partial g_{ab}} = -\frac{1}{8}(-2(R^{Pal}R_{P\ l}^{\ b} + R^{Pbl}R_{P\ l}^{\ a}) + R^P_{ij}R^{ij}_P g^{ab})\sqrt{g}.$$

Using the identities $d_j\sqrt{g} = (1/2)\sqrt{g}g^{kl}g_{kl,j}$ and $g_{kl,j} = g_{kr}\Gamma^r_{lj} + g_{lr}\Gamma^r_{kj}$ we obtain

$$\begin{aligned} hi_\xi d\Theta_{\lambda_{YM}} &= \sqrt{g}(T^{ab}(\xi_{ab} - g_{ab,m}\xi^m) + \nabla_j R^{ji}_P(\xi^P_i - \Gamma^P_{i,m}\xi^m))\omega_0 \\ hdi_\xi\Theta_{\lambda_{YM}} &= d_j\left(-\sqrt{g}\frac{1}{4}R^P_{ki}R^{ki}_P\xi^j + \sqrt{g}R^{ij}_P(\xi^P_i - \Gamma^P_{i,m}\xi^m)\right)\omega_0. \end{aligned}$$

$\diamondsuit$

Therefore we get from Theorem 4.4 the **Einstein-Yang-Mills equations**

$$G^{ab} = T^{ab},\ \nabla_j R^{ji}_P = 0 \tag{4.6}$$

(see e.g. [3, 4, 27, 58]).

Now we will analyze those properties of the action function associated with the Einstein-Yang-Mills Lagrangian λ, which follow from the covariance identity. First we find the chart expression of $C\xi$.

Lemma 4.5. *If a generator of automorphisms ξ on P is locally given by Equation* (3.19)*, then its lift $C\xi$ on the configuration bundle C for the Einstein-Yang-Mills theory is expressed by*

$$\begin{aligned} C\xi = {} & \xi^i\frac{\partial}{\partial x^i} - \left(\frac{\partial\xi^l}{\partial x^i}g_{lj} + \frac{\partial\xi^l}{\partial x^j}g_{il}\right)\frac{\partial}{\partial g_{ij}} \\ & + \left(c^P_{RQ}\xi^R\Gamma^Q_i + \frac{\partial\xi^P}{\partial x^i} - \Gamma^P_j\frac{\partial\xi^j}{\partial x^i}\right)\frac{\partial}{\partial\Gamma^P_i}. \end{aligned} \tag{4.7}$$

Proof: The lift $C\xi$ is given by $C\xi = \left(\frac{d}{dt}C\mathrm{Fl}^\xi_t\right)_0 = \left(\frac{d}{dt}\times_l Z\circ W^1\mathrm{Fl}^\xi_t\right)_0$, where $\times_l Z$ is defined before Theorem 2.4 and we denote by $Z = \mathrm{LMet}\mathbb{R}^n \times \mathrm{L}(\mathbb{R}^n,\mathfrak{g})$ the fiber of the configuration bundle C for the Einstein-Yang-Mills theory. Let $(U_\alpha,\varphi_\alpha)$ be a chart on X at $x = \psi_0(0)$, $\phi_\alpha : p^{-1}(U_\alpha)\to U_\alpha\times G$ a trivialization of P,

$$\begin{gathered} \Phi_\alpha : \tilde{p}^{-1}(U_\alpha)\to U_\alpha\times W^1_nG \cong U_\alpha\times(L^1_n\rtimes T^1_nG); \\ J^1_{(0,e)}\psi\mapsto(x,(J^1_0(t_{-\varphi_\alpha(x)}\circ\varphi_\alpha\circ\psi_0), J^1_0(\mathrm{pr}_2\circ\phi_\alpha\circ\psi_1\circ\varphi^{-1}_\alpha\circ t_{\varphi_\alpha(x)}))) \end{gathered}$$

a local trivialization of W^1P with the inverse

$$\begin{gathered} (x,(J^1_0\alpha, J^1_0a))\mapsto J^1_{(0,e)}(\hat\phi^{-1}_\alpha\circ\tilde\tau_{(\hat\phi_\alpha)_0(x)}\circ(\alpha\circ\mathrm{pr}_1,\mu\circ(a\circ\alpha^{-1}\circ\mathrm{pr}_1,\mathrm{pr}_2))), \\ \hat\phi_\alpha : p^{-1}(U_\alpha)\to\mathbb{R}^n\times G;\ \hat\phi_\alpha = (\varphi_\alpha\times\mathrm{id}_G)\circ\phi_\alpha, \end{gathered}$$

μ is the multiplication in G. We denote by $\tilde{\tau}$ the τ defined on page 13 to distinguish it from the τ defined on page 11 and used below, i.e. $\tilde{\tau}_y = t_y \times \mathrm{id}_G$, e is the unit in G, $\tilde{e}$ is the unit in $W^1_n G$ and let $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times Z$ be a local trivialization of the configuration bundle C. If we write $\hat{e} : \mathbb{R}^n \to G;\ \hat{e}(y) = e$ then we get

$$\Phi_\alpha^{-1}(x, \tilde{e}) = \Phi_\alpha^{-1}(x, (J^1_0 \mathrm{id}_{\mathbb{R}^n}, J^1_0 \hat{e})) = J^1_{(0,e)}(\hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}).$$

So we have

$$\begin{aligned}
\Psi_\alpha \circ [W^1\mathrm{Fl}^\xi_t, \mathrm{id}_Z] \circ \Psi_\alpha^{-1}(x, f) &= (\pi, \mathrm{pr}_2 \circ \Psi_\alpha) \circ [W^1\mathrm{Fl}^\xi_t, \mathrm{id}_Z] \circ [\Phi_\alpha^{-1}(x, \tilde{e}), f] \\
&= (\pi, \mathrm{pr}_2 \circ \Psi_\alpha) \circ [W^1\mathrm{Fl}^\xi_t \circ \Phi_\alpha^{-1}(x, \tilde{e}), f] \\
&= (\tilde{p} \circ W^1\mathrm{Fl}^\xi_t \circ \Phi_\alpha^{-1}(x, \tilde{e}), \mathrm{pr}_2 \circ \Psi_\alpha [W^1\mathrm{Fl}^\xi_t \circ \Phi_\alpha^{-1}(x, \tilde{e}), f]) \\
&= ((W^1\mathrm{Fl}^\xi_t)_0 \circ \tilde{p} \circ \Phi_\alpha^{-1}(x, \tilde{e}), \tau(W^1\mathrm{Fl}^\xi_t \circ \Phi_\alpha^{-1}(x, \tilde{e}), \\
&\qquad \Phi_\alpha^{-1}((W^1\mathrm{Fl}^\xi_t)_0 \circ \tilde{p} \circ \Phi_\alpha^{-1}(x, \tilde{e}))^{-1} \cdot f) \\
&= (\mathrm{Fl}^{\xi_0}_t(x), \tau(\Phi_\alpha^{-1}(\mathrm{Fl}^{\xi_0}_t(x), \tilde{e}), W^1\mathrm{Fl}^\xi_t \circ \Phi_\alpha^{-1}(x, \tilde{e})) \cdot f) \\
&= (\mathrm{Fl}^{\xi_0}_t(x), \tau(J^1_{(0,e)}(\hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(\mathrm{Fl}^{\xi_0}_t(x))}), W^1\mathrm{Fl}^\xi_t \circ J^1_{(0,e)}(\hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)})) \cdot f) \\
&= (\mathrm{Fl}^{\xi_0}_t(x), \tau(J^1_{(0,e)}(\hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(\mathrm{Fl}^{\xi_0}_t(x))}), J^1_{(0,e)}(\mathrm{Fl}^\xi_t \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)})) \cdot f) \\
&= (\mathrm{Fl}^{\xi_0}_t(x), J^1_{(0,e)}(\tilde{\tau}_{-(\hat{\phi}_\alpha)_0(\mathrm{Fl}^{\xi_0}_t(x))} \circ \hat{\phi}_\alpha \circ \mathrm{Fl}^\xi_t \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}) \cdot f). \qquad (4.8)
\end{aligned}$$

Now we can use the fibered coordinates x^i, g_{ij}, Γ^P_i on C induced by the global chart on Z from page 38 and the local trivialization Ψ_α, i.e. the coordinates given by

$$\begin{aligned}
x^i &= x^i \circ \mathrm{pr}_1 \circ \Psi_\alpha = \mathrm{pr}_i \circ \varphi_\alpha \circ \mathrm{pr}_1 \circ \Psi_\alpha, \\
g_{ij} &= g_{ij} \circ \mathrm{pr}_2 \circ \Psi_\alpha,\ \ \Gamma^P_i = \Gamma^P_i \circ \mathrm{pr}_2 \circ \Psi_\alpha, \qquad (4.9)
\end{aligned}$$

to compute the components of the lift $C\xi$. From Equations (4.3), (4.4) and (4.8) we get

$$\begin{aligned}
\bar{x}^i &= x^i([W^1\mathrm{Fl}^\xi_t, \mathrm{id}_Z](\Psi_\alpha^{-1}(x, g, \Gamma))) = x^i(\mathrm{Fl}^{\xi_0}_t(x)), \\
g_{ij}(J^1_{(0,e)}\phi \cdot g) &= \tilde{a}^k_i(J^1_{(0,e)}\phi)\tilde{a}^l_j(J^1_{(0,e)}\phi) g_{kl}(g), \\
\Gamma^P_i(J^1_{(0,e)}\phi \cdot \Gamma) &= \mathrm{A}^P_Q(\mathrm{pr}_2 \circ \beta(J^1_{(0,e)}\phi))(\Gamma^Q_j(\Gamma) + a^Q_j(J^1_{(0,e)}\phi))\tilde{a}^j_i(J^1_{(0,e)}\phi), \\
\phi &= \tilde{\tau}_{-(\hat{\phi}_\alpha)_0(\mathrm{Fl}^\xi_t(x))} \circ \hat{\phi}_\alpha \circ \mathrm{Fl}^\xi_t \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}.
\end{aligned}$$

Now we have to differentiate this at $t = 0$. We get

$$\left(\frac{d}{dt}\bar{x}^i\right)_0 = \xi^i. \qquad (4.10)$$

Since we have

$$
\begin{aligned}
&\tilde{a}_i^k(J^1_{(0,e)}(\tilde{\tau}_{-(\hat{\phi}_\alpha)_0(\mathrm{Fl}_t^{\xi_0}(x))} \circ \hat{\phi}_\alpha \circ \mathrm{Fl}_t^{\xi} \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}))\\
&= D_i((t_{-\varphi_\alpha(\mathrm{Fl}_t^{\xi_0}(x))} \circ \varphi_\alpha \circ \mathrm{Fl}_t^{\xi_0} \circ \varphi_\alpha^{-1} \circ t_{\varphi_\alpha(x)})^{-1})^k(0)\\
&= D_i(x^k \circ \mathrm{Fl}_t^{\xi_0} \circ \varphi_\alpha^{-1})(\varphi_\alpha(\mathrm{Fl}_t^{\xi_0}(x))),
\end{aligned}
$$

where in the last equality we used an identity for the flow of a vector field (see [36] page 78), we get

$$
\begin{aligned}
&\left(\frac{d}{dt}\tilde{a}_i^k(J^1_{(0,e)}(\tilde{\tau}_{-(\hat{\phi}_\alpha)_0(\mathrm{Fl}_t^{\xi_0}(x))} \circ \hat{\phi}_\alpha \circ \mathrm{Fl}_t^{\xi} \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}))\right)_0\\
&= \left(\frac{d}{dt}D_i(x^k \circ \mathrm{Fl}_t^{\xi_0} \circ \varphi_\alpha^{-1})(\varphi_\alpha(\mathrm{Fl}_t^{\xi_0}(x)))\right)_0 = -D_i\xi^k(\varphi_\alpha(x)) = -\frac{\partial \xi^k}{\partial x^i}
\end{aligned}
$$

and so we obtain

$$
\left(\frac{d}{dt}\bar{g}_{ij}\right)_0 = -\left(\frac{\partial \xi^k}{\partial x^i}\delta_j^l + \delta_i^k\frac{\partial \xi^l}{\partial x^j}\right)g_{kl} = -\left(\frac{\partial \xi^k}{\partial x^i}g_{kj} + \frac{\partial \xi^l}{\partial x^j}g_{il}\right), \tag{4.11}
$$

where we denote for simplicity $g_{kl} = g_{kl}(g)$. We write

$$
\begin{aligned}
a(t) &= \mathrm{pr}_2 \circ \beta(J^1_{(0,e)}(\tilde{\tau}_{-(\hat{\phi}_\alpha)_0(\mathrm{Fl}_t^{\xi_0}(x))} \circ \hat{\phi}_\alpha \circ \mathrm{Fl}_t^{\xi} \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}))\\
&= \mathrm{pr}_2 \circ \tilde{\tau}_{-(\hat{\phi}_\alpha)_0(\mathrm{Fl}_t^{\xi_0}(x))} \circ \hat{\phi}_\alpha \circ \mathrm{Fl}_t^{\xi} \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}(0,e) = \mathrm{pr}_2 \circ \phi_\alpha \circ \mathrm{Fl}_t^{\xi} \circ \phi_\alpha^{-1}(x,e).
\end{aligned}
$$

Then the computation

$$
\begin{aligned}
\left(\frac{d}{dt}A_Q^P(a(t))e_P\right)_0 &= \left(\frac{d}{dt}\mathrm{Ad}(a(t))e_Q\right)_0 = T_e\mathrm{Ad}\left(\frac{d}{dt}a(t)\right)_0 e_Q\\
&= \mathrm{ad}\left(\frac{d}{dt}a(t)\right)_0 e_Q = \mathrm{ad}(\xi^R(x)e_R)e_Q = [\xi^R(x)e_R, e_Q]\\
&= \xi^R(x)[e_R, e_Q] = \xi^R(x)c_{RQ}^P e_P,
\end{aligned}
$$

where we have used Definition (3.18), shows that

$$
\left(\frac{d}{dt}A_Q^P(a(t))\right)_0 = c_{RQ}^P\xi^R,
$$

where we write for simplicity $\xi^R = \xi^R(x)$. Let us denote $a^P = \mathrm{pr}_P \circ \exp^{-1}$ the canonical coordinates on a neighborhood of the unit in G corresponding to the basis e_P in $\mathfrak{g}$, i.e. we have $\left(\frac{\partial}{\partial a^P}\right)_e = e_P$. Using the identities for the multiplication μ and the inversion ν in the group G

$$
T_{(a,b)}\mu \cdot (X_a, Y_b) = T_a(\rho_b)\cdot X_a + T_b(\lambda_a)Y_b,\ T_a\nu = -T_e(\lambda_{a^{-1}})\cdot T_a(\rho_{a^{-1}})
$$

for all $a, b \in G$ and if realize that for the left translation λ and the right translation ρ the relation $\lambda_e = \rho_e = \mathrm{id}_G$ holds, then we can compute

$$\left(\frac{d}{dt} a_j^Q (J^1_{(0,e)}(\tilde{\tau}_{-(\hat{\phi}_\alpha)_0(\mathrm{Fl}_t^{\xi_0}(x))} \circ \hat{\phi}_\alpha \circ \mathrm{Fl}_t^\xi \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}))\right)_0$$
$$= \left(\frac{d}{dt} D_j (a^{-1}(t)(\mathrm{pr}_2 \circ \phi_\alpha \circ \mathrm{Fl}_t^\xi \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}(\ ,e)))^Q(0)\right)_0$$
$$= D_j \left(\frac{d}{dt} a^Q \circ \mu \circ (\nu \circ a(t), \mathrm{pr}_2 \circ \phi_\alpha \circ \mathrm{Fl}_t^\xi \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}(\ ,e))\right)_0 (0)$$
$$= D_j \left(- \left(\frac{d}{dt} a^Q \circ a(t)\right)_0 + \left(\frac{d}{dt} a^Q \circ \mathrm{pr}_2 \circ \phi_\alpha \circ \mathrm{Fl}_t^\xi \circ \hat{\phi}_\alpha^{-1} \circ \tilde{\tau}_{(\hat{\phi}_\alpha)_0(x)}(\ ,e)\right)_0 \right)(0)$$
$$= D_j \left(\left(\frac{d}{dt} a^Q \circ \mathrm{pr}_2 \circ \phi_\alpha \circ \mathrm{Fl}_t^\xi \circ \phi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times \mathrm{id}_G) \circ (t_{\varphi_\alpha(x)} \times \mathrm{id}_G)(\ ,e)\right)_0 \right)(0)$$
$$= D_j(\xi^Q \circ \varphi_\alpha^{-1} \circ t_{\varphi_\alpha(x)}(\ ))(0) = D_j(\xi^Q(\varphi_\alpha^{-1}(\ )))(\varphi_\alpha(x)) = \frac{\partial \xi^Q}{\partial x^j}.$$

Thus we get

$$\left(\frac{d}{dt}\bar{\Gamma}_i^P\right)_0 = c^P_{RQ}\xi^R\Gamma_j^Q\delta_i^j + \delta_Q^P \frac{\partial \xi^Q}{\partial x^j}\delta_i^j - \delta_Q^P \Gamma_j^Q \frac{\partial \xi^j}{\partial x^i}$$
$$= c^P_{RQ}\xi^R\Gamma_i^Q + \frac{\partial \xi^P}{\partial x^i} - \Gamma_j^P \frac{\partial \xi^j}{\partial x^i}, \tag{4.12}$$

where we denote for simplicity $\Gamma_j^Q = \Gamma_j^Q(\Gamma)$. From Equations (4.10), (4.11) and (4.12) we immediately obtain Equation (4.7) and this finishes the proof.

$\diamondsuit$

The gravitational part of the lift $C\xi$ agrees with [35] and the Yang-Mills part with [19] and [20].

The notion of the jet prolongation of a vector field is standard (see [28] and [25]). If (x^i, y^σ) are fibered coordinates of a fibered manifold Y, then the 1-jet prolongation of a projectable vector field $\zeta = \zeta^i(x)(\partial/\partial x^i) + \zeta^\sigma(x,y)(\partial/\partial y^\sigma)$ is

$$J^1\zeta = \zeta^i \frac{\partial}{\partial x^i} + \zeta^\sigma \frac{\partial}{\partial y^\sigma} + \left(\frac{\partial \zeta^\sigma}{\partial x^i} + \frac{\partial \zeta^\sigma}{\partial y^\rho} y_i^\rho - \frac{\partial \zeta^j}{\partial x^i} y_j^\sigma\right)\frac{\partial}{\partial y_i^\sigma}. \tag{4.13}$$

Lemma 4.6. *If a generator of automorphisms ξ on P is locally given by Equation* (3.19)*, then its lift $J^1C\xi$ on the first jet prolongation of the configuration* *bundle C for the Einstein-Yang-Mills theory is expressed by*

$$\begin{aligned} J^1C\xi = {} & \xi^i \frac{\partial}{\partial x^i} - \left(\frac{\partial \xi^l}{\partial x^i} g_{lj} + \frac{\partial \xi^l}{\partial x^j} g_{il} \right) \frac{\partial}{\partial g_{ij}} \\ & + \left(c^P_{RQ} \xi^R \Gamma^Q_i + \frac{\partial \xi^P}{\partial x^i} - \Gamma^P_j \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial \Gamma^P_i} \\ & - \left(\frac{\partial \xi^l}{\partial x^i \partial x^k} g_{lj} + \frac{\partial \xi^l}{\partial x^j \partial x^k} g_{il} + \frac{\partial \xi^l}{\partial x^i} g_{lj,k} + \frac{\partial \xi^l}{\partial x^j} g_{il,k} + g_{ij,l} \frac{\partial \xi^l}{\partial x^k} \right) \frac{\partial}{\partial g_{ij,k}} \\ & + \left(c^P_{RQ} \frac{\partial \xi^R}{\partial x^j} \Gamma^Q_i + \frac{\partial \xi^P}{\partial x^i \partial x^j} - \Gamma^P_k \frac{\partial \xi^k}{\partial x^i \partial x^j} + c^P_{RQ} \xi^R \Gamma^Q_{i,j} \right. \\ & \left. - \frac{\partial \xi^k}{\partial x^i} \Gamma^P_{k,j} - \Gamma^P_{i,l} \frac{\partial \xi^l}{\partial x^j} \right) \frac{\partial}{\partial \Gamma^P_{i,j}}. \end{aligned} \tag{4.14}$$

Proof: This follows immediately from Lemma 4.5 and Equation (4.13).

♢

Now we can proceed to the discussion of the first variation formula of the Hilbert-Yang-Mills Lagrangian for the induced variations. We denote by

$$\mathcal{E}^{ab} = \sqrt{g}(-G^{ab} + T^{ab}), \ \mathcal{F}^i_P = \sqrt{g} \nabla_j R^{ji}_P$$

the Euler-Lagrange expressions of the Hilbert-Yang-Mills Lagrangian, and by $\xi^J_V = \xi^J - \Gamma^J_i \xi^i$ the components of the vertical part ξ_V of the generator of automorphisms ξ with respect to the principal connection. We define the **Komar-Yang-Mills superpotential** by

$$\nu_\xi = \frac{1}{2} \sqrt{g} (\nabla^{[i} \xi^{j]} - R^{ij}_J \xi^J_V) \omega_{ij},$$

where ∇ denotes the covariant derivative with respect to the Levi-Civita connection. The first term in the Komar-Yang-Mills superpotential is the so called **Komar potential** [26].

The following basic theorem clarifies the structure of the currents associated with vector fields on the underlying principal bundle P.

Theorem 4.7. *For the principal Lepage equivalent of the Hilbert-Yang-Mills Lagrangian Θ_λ the Euler-Lagrange term has a chart expression*

$$\begin{aligned} h i_{J^1C\xi} d\Theta_\lambda = {} & -\mathcal{E}^{ab}(g_{ac} d_b \xi^c + g_{bc} d_a \xi^c + g_{ab,c} \xi^c)\omega_0 \\ & + \mathcal{F}^i_P (c^P_{RQ} \xi^R \Gamma^Q_i + d_i \xi^P - \Gamma^P_j d_i \xi^j - \Gamma^P_{i,m} \xi^m)\omega_0. \end{aligned}$$

The current has a chart expression

$$\begin{aligned} i_{J^1C\xi}\Theta_\lambda = {} & (2\mathcal{E}^{ib} g_{bj} \xi^j - \mathcal{F}^i_J \xi^J_V)\omega_i + h d\nu_\xi \\ & + \sqrt{g} \xi^i [(g^{jc} g^{ab} - g^{bc} g^{ja})(\eta_{ab,c} - \Gamma^d_{ab} \eta_{dc}) + R^{jk}_P \eta^P_k] \wedge \omega_{ij}. \end{aligned} \tag{4.15}$$

Proof: We get the Euler-Lagrange term from Theorem 4.4 where we substitute from Lemma 4.6

$$\xi_{ij} = -\left(\frac{\partial \xi^l}{\partial x^i} g_{lj} + \frac{\partial \xi^l}{\partial x^j} g_{il}\right), \ \xi^P_i = c^P_{RQ}\xi^R\Gamma^Q_i + \frac{\partial \xi^P}{\partial x^i} - \Gamma^P_j \frac{\partial \xi^j}{\partial x^i},$$

and we use the fact that $\partial\xi^l/\partial x^i = d_i\xi^l$, $\partial\xi^P/\partial x^i = d_i\xi^P$, because ξ^l and ξ^P depend only on x.

Let us denote by

$$J^1C_g\xi = \xi^i\frac{\partial}{\partial x^i} - \left(\frac{\partial \xi^l}{\partial x^i} g_{lj} + \frac{\partial \xi^l}{\partial x^j} g_{il}\right)\frac{\partial}{\partial g_{ij}}$$
$$-\left(\frac{\partial \xi^l}{\partial x^i\partial x^k} g_{lj} + \frac{\partial \xi^l}{\partial x^j\partial x^k} g_{il} + \frac{\partial \xi^l}{\partial x^i} g_{lj,k} + \frac{\partial \xi^l}{\partial x^j} g_{il,k} + g_{ij,l}\frac{\partial \xi^l}{\partial x^k}\right)\frac{\partial}{\partial g_{ij,k}}$$

the gravitational part of the lift $J^1C\xi$ from Equation (4.14). Then we can write

$$i_{J^1C\xi}\Theta_\lambda = i_{J^1C_g\xi}\Theta_{\lambda_H} + i_{J^1C\xi}\Theta_{\lambda_{YM}}.$$

The gravitational term $i_{J^1C_g\xi}\Theta_{\lambda_H}$ was computed in [35]

$$i_{J^1C_g\xi}\Theta_{\lambda_H} = w^i\omega_i + \sqrt{g}\xi^i(g^{jc}g^{ab} - g^{bc}g^{ja})(\eta_{ab,c} - \Gamma^d_{ab}\eta_{dc}) \wedge \omega_{ij},$$
$$w^i = L^i_j\xi^j + L^{ip}_j d_p\xi^j + L^{ipq}_j d_p d_q\xi^j,$$
$$L^i_m = \sqrt{g}(R\delta^i_m - (g^{rq}g^{si} - g^{sr}g^{qi})g_{pk}\Gamma^p_{rs}\Gamma^k_{qm} - g^{si}\Gamma^p_{rs}\Gamma^r_{pm}$$
$$+g^{sr}\Gamma^p_{rs}\Gamma^i_{pm} - (g^{ps}g^{iq} - g^{is}g^{qp})g_{pq,sm}),$$
$$L^{ip}_m = \sqrt{g}(\delta^i_m g^{sr}\Gamma^p_{rs} + g^{ip}\Gamma^q_{qm} - 2g^{ps}\Gamma^i_{sm}),$$
$$L^{ipq}_m = -\frac{1}{2}\sqrt{g}(2\delta^i_m g^{pq} - \delta^p_m g^{iq} - \delta^q_m g^{ip}).$$

For the Yang-Mills term $i_{J^1C\xi}\Theta_{\lambda_{YM}}$ we get

$$i_{J^1C\xi}\Theta_{\lambda_{YM}} = -\sqrt{g}\frac{1}{4}R^P_{kj}R^{kj}_P\xi^i\omega_i + \sqrt{g}R^{ij}_P(c^P_{RQ}\xi^R\Gamma^Q_i + d_i\xi^P - \Gamma^P_k d_i\xi^k)\omega_j$$
$$-\sqrt{g}R^{ij}_P\xi^k\eta^P_i \wedge \omega_{jk}.$$

Thus for the whole current we obtain

$$i_{J^1C\xi}\Theta_\lambda = [w^i - \sqrt{g}\frac{1}{4}R^P_{kj}R^{kj}_P\xi^i + \sqrt{g}R^{ji}_P(c^P_{RQ}\xi^R\Gamma^Q_j + d_j\xi^P - \Gamma^P_k d_j\xi^k)]\omega_i$$
$$+\sqrt{g}\xi^i[(g^{jc}g^{ab} - g^{bc}g^{ja})(\eta_{ab,c} - \Gamma^d_{ab}\eta_{dc}) + R^{kj}_P\eta^P_k] \wedge \omega_{ij}.$$

Now we want to write the current as in Equation (4.15). The first variation formula implies that the covariance identity $\partial_{J^2C\xi}\lambda = 0$ is equivalent to $hi_{J^1C\xi}d\Theta_\lambda + hdi_{J^1C\xi}\Theta_\lambda = 0$. Since the contact part of the current is annihilated by the horizontalization h, we get

$$-\mathcal{E}^{ab}(g_{ac}d_b\xi^c + g_{bc}d_a\xi^c + g_{ab,c}\xi^c)$$
$$+\mathcal{F}^i_P(c^P_{RQ}\xi^R\Gamma^Q_i + d_i\xi^P - \Gamma^P_j d_i\xi^j - \Gamma^P_{i,m}\xi^m) + d_iv^i = 0, \tag{4.16}$$
$$v^i = w^i - \sqrt{g}\frac{1}{4}R^P_{kj}R^{kj}_P\xi^i + \sqrt{g}R^{ji}_P(c^P_{RQ}\xi^R\Gamma^Q_j + d_j\xi^P - \Gamma^P_k d_j\xi^k).$$

Now we use the same strategy as in [35] and write v^i in the form

$$v^i = M^i_j \xi^j + M^{ip}_j d_p \xi^j + M^{ipq}_j d_p d_q \xi^j + N^i_R \xi^R + N^{ij}_R d_j \xi^R, \tag{4.17}$$
$$M^i_j = L^i_j - \sqrt{g}\frac{1}{4} R^P_{kl} R^{kl}_P \delta^i_j,\ M^{ip}_j = L^{ip}_j - \sqrt{g} R^{pi}_Q \Gamma^Q_j,\ M^{ipq}_j = L^{ipq}_j$$
$$N^i_R = \sqrt{g} R^{ji}_P c^P_{RQ} \Gamma^Q_j,\ N^{ij}_R = \sqrt{g} R^{ji}_R.$$

If we substitute Equation (4.17) into Equation (4.16) we get

$$\begin{aligned}
&-\mathcal{E}^{ab}(2g_{bc} d_a \xi^c + g_{ab,c}\xi^c) + \mathcal{F}^i_P (c^P_{RQ}\xi^R \Gamma^Q_i + d_i \xi^P - \Gamma^P_j d_i \xi^j - \Gamma^P_{i,m}\xi^m)\\
&+ d_i M^i_j \xi^j + (M^p_j + d_i M^{ip}_j) d_p \xi^j + (M^{qp}_j + d_i M^{ipq}_j) d_q d_p \xi^j + M^{ipq}_j d_i d_q d_p \xi^j\\
&+ d_i N^i_J \xi^J + (N^p_J + d_i N^{ip}_J) d_p \xi^J + N^{qp}_J d_q d_p \xi^J\\
&= (-\mathcal{E}^{ab} g_{ab,j} - \mathcal{F}^i_P \Gamma^P_{i,j} + d_i M^i_j)\xi^j + (-2\mathcal{E}^{pb} g_{bj} - \mathcal{F}^p_R \Gamma^R_j + M^p_j + d_i M^{ip}_j) d_p \xi^j\\
&+ (M^{qp}_j + d_i M^{ipq}_j) d_q d_p \xi^j + M^{ipq}_j d_i d_q d_p \xi^j\\
&+ (\mathcal{F}^i_P c^P_{JQ} \Gamma^Q_i + d_i N^i_J)\xi^J + (\mathcal{F}^p_J + N^p_J + d_i N^{ip}_J) d_p \xi^J + N^{qp}_J d_q d_p \xi^J = 0.
\end{aligned}$$

The last equation holds for every generator of automorphisms iff the following identities hold

$$-\mathcal{E}^{ab} g_{ab,j} - \mathcal{F}^i_P \Gamma^P_{i,j} + d_i M^i_j = 0, \tag{4.18}$$
$$-\mathcal{E}^{pb} 2 g_{bj} - \mathcal{F}^p_R \Gamma^R_j + M^p_j + d_i M^{ip}_j = 0, \tag{4.19}$$
$$\frac{1}{2}(M^{qp}_j + M^{pq}_j) + d_i M^{ipq}_j = 0, \tag{4.20}$$
$$M^{ipq}_j + M^{qip}_j + M^{pqi}_j = 0, \tag{4.21}$$
$$\mathcal{F}^i_P c^P_{JQ} \Gamma^Q_i + d_i N^i_J = 0, \tag{4.22}$$
$$\mathcal{F}^p_J + N^p_J + d_i N^{ip}_J = 0, \tag{4.23}$$
$$N^{qp}_J + N^{pq}_J = 0. \tag{4.24}$$

Since $N^{ij}_R = \sqrt{g} R^{ji}_R$ and R^{ji}_P is antisymmetric in the upper indices, the last identity is trivially satisfied. We have

$$\begin{aligned}
&M^i_j \xi^j + M^{ip}_j d_p \xi^j + M^{ipq}_j d_p d_q \xi^j\\
&= M^i_j \xi^j + d_p(M^{ip}_j \xi^j) - d_p M^{ip}_j \xi^j + d_q(M^{ipq}_j d_p \xi^j) - d_q M^{ipq}_j d_p \xi^j\\
&= M^i_j \xi^j - d_p M^{ip}_j \xi^j + d_p d_q M^{ipq}_j \xi^j + d_p(M^{ip}_j \xi^j) + d_q(M^{ipq}_j d_p \xi^j) - d_p(d_q M^{ipq}_j \xi^j)\\
&= (M^i_j - d_p M^{ip}_j + d_p d_q M^{ipq}_j)\xi^j + d_p((M^{ip}_j - d_q M^{ipq}_j)\xi^j + M^{ipq}_j d_q \xi^j)
\end{aligned}$$

We define a $(n-2)$-form η_ξ by

$$\eta_\xi = \frac{1}{2}((M^{ip}_j - d_q M^{ipq}_j)\xi^j + M^{ipq}_j d_q \xi^j)\omega_{ip}.$$

Since $dx^k \wedge \omega_{ip} = -\delta_i^k \omega_p + \delta_p^k \omega_i$ we have

$$\begin{aligned} hd\eta_\xi &= \frac{1}{2} d_k((M_j^{ip} - d_q M_j^{ipq})\xi^j + M_j^{ipq} d_q \xi^j) dx^k \wedge \omega_{ip} \\ &= -\frac{1}{2} d_i((M_j^{ip} - d_q M_j^{ipq})\xi^j + M_j^{ipq} d_q \xi^j)\omega_p \\ &\quad + \frac{1}{2} d_p((M_j^{ip} - d_q M_j^{ipq})\xi^j + M_j^{ipq} d_q \xi^j)\omega_i \\ &= \frac{1}{2} d_p((M_j^{ip} - M_j^{pi} - d_q(M_j^{ipq} - M_j^{piq}))\xi^j + (M_j^{ipq} - M_j^{piq}) d_q \xi^j)\omega_i. \end{aligned}$$

Thus we get

$$\begin{aligned} &(M_j^i \xi^j + M_j^{ip} d_p \xi^j + M_j^{ipq} d_p d_q \xi^j)\omega_i = (M_j^i - d_p M_j^{ip} + d_p d_q M_j^{ipq})\xi^j \omega_i \\ &+ \frac{1}{2} d_p((M_j^{ip} + M_j^{pi} - d_q(M_j^{ipq} + M_j^{piq}))\xi^j + (M_j^{ipq} + M_j^{piq}) d_q \xi^j)\omega_i \\ &+ \frac{1}{2} d_p((M_j^{ip} - M_j^{pi} - d_q(M_j^{ipq} - M_j^{piq}))\xi^j + (M_j^{ipq} - M_j^{piq}) d_q \xi^j)\omega_i \\ &= (M_j^i - d_p M_j^{ip} + d_p d_q M_j^{ipq})\xi^j \omega_i \\ &+ \frac{1}{2} d_p((M_j^{ip} + M_j^{pi} - d_q(M_j^{ipq} + M_j^{piq}))\xi^j + (M_j^{ipq} + M_j^{piq}) d_q \xi^j)\omega_i + hd\eta_\xi. \end{aligned}$$

Taking into account Equations (4.20) and (4.21), we have

$$\begin{aligned} &\frac{1}{2}(M_j^{ip} + M_j^{pi} - d_q(M_j^{ipq} + M_j^{piq}))\xi^j + (M_j^{ipq} + M_j^{piq}) d_q \xi^j \\ &= -d_s M_j^{sip} \xi^j + \frac{1}{2} d_q M_j^{qip} \xi^j - \frac{1}{2} M_j^{qip} d_q \xi^j = -\frac{1}{2} d_q(M_j^{qip} \xi^j). \end{aligned}$$

If we set

$$\mu_\xi = \frac{1}{6} d_q(M_j^{ipq} \xi^j)\omega_{ip},$$

then

$$\begin{aligned} hd\mu_\xi &= \frac{1}{6} d_k d_q(M_j^{ipq} \xi^j) dx^k \wedge \omega_{ip} = -\frac{1}{6} d_i d_q(M_j^{ipq} \xi^j)\omega_p + \frac{1}{6} d_p d_q(M_j^{ipq} \xi^j)\omega_i \\ &= \frac{1}{6} d_p d_q((M_j^{ipq} - M_j^{piq})\xi^j)\omega_i = -\frac{1}{6} d_p d_q((M_j^{qip} + M_j^{pqi} + M_j^{piq})\xi^j)\omega_i \\ &= -\frac{1}{2} d_p d_q(M_j^{qip} \xi^j)\omega_i. \end{aligned}$$

Therefore we obtain

$$(M_j^i \xi^j + M_j^{ip} d_p \xi^j + M_j^{ipq} d_p d_q \xi^j)\omega_i = (M_j^i - d_p M_j^{ip} + d_p d_q M_j^{ipq})\xi^j \omega_i + hd\tau_\xi,$$

where

$$\begin{aligned}
\tau_\xi &= \mu_\xi + \eta_\xi \\
&= \frac{1}{6} d_q(M_j^{ipq}\xi^j)\omega_{ip} + \frac{1}{2}((M_j^{ip} - d_q M_j^{ipq})\xi^j + M_j^{ipq} d_q\xi^j)\omega_{ip} \\
&= \frac{1}{6}(3M_j^{ip}\xi^j - 3d_q M_j^{ipq}\xi^j + 3M_j^{ipq} d_q\xi^j + d_q M_j^{ipq}\xi^j + M_j^{ipq} d_q\xi^j)\omega_{ip} \\
&= \frac{1}{6}(3M_j^{ip}\xi^j - 2d_q M_j^{ipq}\xi^j + 4M_j^{ipq} d_q\xi^j)\omega_{ip}.
\end{aligned}$$

Taking into account Equations (4.19), (4.20) and (4.21), we have

$$\begin{aligned}
M_j^i - d_p M_j^{ip} + d_p d_q M_j^{ipq} &= \mathcal{E}^{ib} 2g_{bj} + \mathcal{F}_R^i \Gamma_j^R - d_p(M_j^{pi} + M_j^{ip} - d_q M_j^{ipq}) \\
&= \mathcal{E}^{ib} 2g_{bj} + \mathcal{F}_R^i \Gamma_j^R + d_p d_q(M_j^{qip} + M_j^{piq} + M_j^{ipq}) = \mathcal{E}^{ib} 2g_{bj} + \mathcal{F}_R^i \Gamma_j^R.
\end{aligned}$$

Thus the first part ("M" part) of the current has the form

$$(M_j^i\xi^j + M_j^{ip} d_p\xi^j + M_j^{ipq} d_p d_q\xi^j)\omega_i = (\mathcal{E}^{ib} 2g_{bj} + \mathcal{F}_R^i \Gamma_j^R)\xi^j\omega_i + hd\tau_\xi.$$

We can proceed similarly for the second part ("N" part) of the current with an obvious modification - the highest M_j^{ipq} can be replaced by zero and we have to use the identities (4.23) and (4.24). So we define κ_ξ instead of η_ξ

$$\kappa_\xi = \frac{1}{2} N_J^{ip}\xi^J\omega_{ip}.$$

Equation (4.24) implies that we need not introduce any form corresponding to μ_ξ. From Equations (4.23) and (4.24) we get

$$N_J^i - d_p N_J^{ip} = -\mathcal{F}_J^i - d_p(N_J^{pi} + N_J^{ip}) = -\mathcal{F}_J^i.$$

Thus for the whole current we have

$$\begin{aligned}
i_{J^1C\xi}\Theta_\lambda &= (\mathcal{E}^{ib} 2g_{bj} + \mathcal{F}_R^i \Gamma_j^R)\xi^j\omega_i + hd\tau - \mathcal{F}_J^i\xi^J\omega_i + hd\kappa_\xi \\
&+ \sqrt{g}\xi^i[(g^{jc}g^{ab} - g^{bc}g^{ja})(\eta_{ab,c} - \Gamma_{ab}^d\eta_{dc}) + R_P^{kj}\eta_k^P] \wedge \omega_{ij} \\
&= [(2\mathcal{E}^{ib} g_{bj}\xi^j - \mathcal{F}_J^i(\xi^J - \Gamma_j^J\xi^j)]\omega_i + hd\nu_\xi \\
&+ \sqrt{g}\xi^i[(g^{jc}g^{ab} - g^{bc}g^{ja})(\eta_{ab,c} - \Gamma_{ab}^d\eta_{dc}) + R_P^{kj}\eta_k^P] \wedge \omega_{ij},
\end{aligned}$$

where

$$\begin{aligned}
\nu_\xi = \tau_\xi + \kappa_\xi &= \frac{1}{6}(3M_j^{ip}\xi^j - 2d_q M_j^{ipq}\xi^j + 4M_j^{ipq} d_q\xi^j + 3N_J^{ip}\xi^J)\omega_{ip} \\
&= \frac{1}{6}(3L_j^{ip}\xi^j - 2d_q L_j^{ipq}\xi^j + 4L_j^{ipq} d_q\xi^j - 3\sqrt{g}R_Q^{pi}\Gamma_j^Q\xi^j + 3N_J^{ip}\xi^J)\omega_{ip}.
\end{aligned}$$

This is nearly the required expression for the current. To finish this proof it suffices to show that the superpotential ν_ξ has the demanded form. But from the identities

$$d_q\sqrt{g} = \sqrt{g}\Gamma_{qr}^r, \quad d_q g^{ip} = -g^{kp}\Gamma_{kq}^i - g^{ik}\Gamma_{kq}^p$$

we get

$$d_q L_j^{ipq} = -\frac{1}{2} d_q \sqrt{g}(2\delta_j^i g^{pq} - \delta_j^p g^{iq} - \delta_j^q g^{ip})$$

$$-\frac{1}{2}\sqrt{g}(2\delta_j^i d_q g^{pq} - \delta_j^p d_q g^{iq} - \delta_j^q d_q g^{ip}) = -\frac{1}{2}\sqrt{g}\Gamma_{qr}^r(2\delta_j^i g^{pq} - \delta_j^p g^{iq} - \delta_j^q g^{ip})$$

$$+\frac{1}{2}\sqrt{g}(2\delta_j^i(g^{kq}\Gamma_{kq}^p + g^{pk}\Gamma_{kq}^q) - \delta_j^p(g^{kq}\Gamma_{kq}^i + g^{ik}\Gamma_{kq}^q) - \delta_j^q(g^{kp}\Gamma_{kq}^i + g^{ik}\Gamma_{kq}^p))$$

$$= -\frac{1}{2}\sqrt{g}\Gamma_{qr}^r(2\delta_j^i g^{pq} - \delta_j^p g^{iq}) + \frac{1}{2}\sqrt{g} g^{ip}\Gamma_{jr}^r$$

$$+\frac{1}{2}\sqrt{g}(2\delta_j^i(g^{kq}\Gamma_{kq}^p + g^{pk}\Gamma_{kq}^q) - \delta_j^p(g^{kq}\Gamma_{kq}^i + g^{ik}\Gamma_{kq}^q)) - \frac{1}{2}\sqrt{g}(g^{kp}\Gamma_{kj}^i + g^{ik}\Gamma_{kj}^p)$$

$$= \frac{1}{2}\sqrt{g}(2\delta_j^i(g^{kq}\Gamma_{kq}^p + g^{pk}\Gamma_{kq}^q - g^{pq}\Gamma_{qr}^r) - \delta_j^p(g^{kq}\Gamma_{kq}^i + g^{ik}\Gamma_{kq}^q - g^{iq}\Gamma_{qr}^r))$$

$$+\frac{1}{2}\sqrt{g}(g^{ip}\Gamma_{jr}^r - g^{kp}\Gamma_{kj}^i - g^{ik}\Gamma_{kj}^p)$$

$$= \frac{1}{2}\sqrt{g}(2\delta_j^i g^{kq}\Gamma_{kq}^p - \delta_j^p g^{kq}\Gamma_{kq}^i) + \frac{1}{2}\sqrt{g}(g^{ip}\Gamma_{jr}^r - g^{kp}\Gamma_{kj}^i - g^{ik}\Gamma_{kj}^p).$$

Thus we have

$$3L_j^{ip}\xi^j - 2d_q L_j^{ipq}\xi^j + 4L_j^{ipq}d_q\xi^j = 3\sqrt{g}(\delta_j^i g^{sr}\Gamma_{rs}^p + g^{ip}\Gamma_{qj}^q - 2g^{ps}\Gamma_{sj}^i)\xi^j$$

$$-\sqrt{g}(2\delta_j^i g^{kq}\Gamma_{kq}^p - \delta_j^p g^{kq}\Gamma_{kq}^i)\xi^j - \sqrt{g}(g^{ip}\Gamma_{jr}^r - g^{kp}\Gamma_{kj}^i - g^{ik}\Gamma_{kj}^p)\xi^j$$

$$-2\sqrt{g}(2\delta_j^i g^{pq} - \delta_j^p g^{iq} - \delta_j^q g^{ip})d_q\xi^j = \sqrt{g}(3\delta_j^i g^{sr}\Gamma_{rs}^p + 3g^{ip}\Gamma_{qj}^q - 6g^{ps}\Gamma_{sj}^i$$

$$-2\delta_j^i g^{kq}\Gamma_{kq}^p + \delta_j^p g^{kq}\Gamma_{kq}^i - g^{ip}\Gamma_{jr}^r + g^{kp}\Gamma_{kj}^i + g^{ik}\Gamma_{kj}^p)\xi^j$$

$$-2\sqrt{g}(2\delta_j^i g^{pq} - \delta_j^p g^{iq} - \delta_j^q g^{ip})d_q\xi^j$$

$$= \sqrt{g}(\delta_j^i(3g^{sr}\Gamma_{rs}^p - 2g^{kq}\Gamma_{kq}^p) + \delta_j^p g^{kq}\Gamma_{kq}^i + 3g^{ip}\Gamma_{qj}^q - 6g^{ps}\Gamma_{sj}^i$$

$$-g^{ip}\Gamma_{jr}^r + g^{kp}\Gamma_{kj}^i + g^{ik}\Gamma_{kj}^p)\xi^j - 2\sqrt{g}(2\delta_j^i g^{pq} - \delta_j^p g^{iq} - \delta_j^q g^{ip})d_q\xi^j$$

$$= \sqrt{g}(\delta_j^i g^{sr}\Gamma_{rs}^p + \delta_j^p g^{kq}\Gamma_{kq}^i + 2g^{ip}\Gamma_{qj}^q - 5g^{ps}\Gamma_{sj}^i + g^{ik}\Gamma_{kj}^p)\xi^j$$

$$-2\sqrt{g}(2\delta_j^i g^{pq} - \delta_j^p g^{iq} - \delta_j^q g^{ip})d_q\xi^j.$$

Antisymmetrizing the last expression in i and p, we get

$$\frac{1}{2}[\sqrt{g}(-5g^{ps}\Gamma_{sj}^i + 5g^{is}\Gamma_{sj}^p + g^{ik}\Gamma_{kj}^p - g^{pk}\Gamma_{kj}^i)\xi^j$$

$$-2\sqrt{g}(2\delta_j^i g^{pq} - 2\delta_j^p g^{iq} - \delta_j^p g^{iq} + \delta_j^i g^{pq})d_q\xi^j]$$

$$= \frac{1}{2}\sqrt{g}(-6g^{ps}\Gamma_{sj}^i + 6g^{is}\Gamma_{sj}^p)\xi^j - 3\sqrt{g}(\delta_j^i g^{pq} - \delta_j^p g^{iq})d_q\xi^j$$

$$= 3\sqrt{g}(-g^{ps}\Gamma_{sj}^i\xi^j + g^{is}\Gamma_{sj}^p\xi^j - g^{pq}d_q\xi^i + g^{iq}d_q\xi^p)$$

$$= 3\sqrt{g}(g^{is}(d_s\xi^p + \Gamma_{sj}^p\xi^j) - g^{ps}(d_s\xi^i + \Gamma_{sj}^i\xi^j))$$

$$= 3\sqrt{g}(g^{is}\nabla_s\xi^p - g^{ps}\nabla_s\xi^i) = 3\sqrt{g}\nabla^{[i}\xi^{p]}.$$

Therefore we finally obtain

$$\nu_\xi = \frac{1}{6}(3\sqrt{g}\nabla^{[i}\xi^{p]} - 3\sqrt{g}R^{pi}_Q\Gamma^Q_j\xi^j + 3N^{ip}_J\xi^J)\omega_{ip}$$
$$= \frac{1}{2}\sqrt{g}(\nabla^{[i}\xi^{p]} - R^{pi}_Q\Gamma^Q_j\xi^j + R^{pi}_J\xi^J)\omega_{ip} = \frac{1}{2}\sqrt{g}(\nabla^{[i}\xi^{p]} - R^{ip}_J(\xi^J - \Gamma^J_j\xi^j))\omega_{ip}$$
$$= \frac{1}{2}\sqrt{g}(\nabla^{[i}\xi^{p]} - R^{ip}_J\xi^J_V)\omega_{ip}.$$

This finishes the proof.

$\diamondsuit$

Since we could add the contact part of $d\nu_\xi$ to the last term, we see that the current can be written in the form

$$i_{J^1C\xi}\Theta_\lambda = (2\mathcal{E}^i_j\xi^j - \mathcal{F}^i_J\xi^J_V)\omega_i + d\nu_\xi + \eta, \tag{4.25}$$

where $\mathcal{E}^i_j = \mathcal{E}^{ib}g_{bj}$. The first term in (4.25) vanishes along solutions of the Euler-Lagrange (i.e. Einstein-Yang-Mills) equations, the second term is exact, the third term η is contact. For every section γ of C we have

$$J^1\gamma^* i_{J^1C\xi}\Theta_\lambda = [2(\mathcal{E}^i_j \circ J^2\gamma)\xi^j - (\mathcal{F}^i_J \circ J^2\gamma)\xi^J_V]\omega_i + dj^1\gamma^*\nu_\xi. \tag{4.26}$$

If γ is a solution of the Euler-Lagrange equations, i.e. if $\mathcal{E}^i_j \circ J^2\gamma = 0$, $\mathcal{F}^i_J \circ J^2\gamma = 0$, then $J^1\gamma^* i_{J^1C\xi}\Theta_\lambda$ is an exact form.

Corollary 4.8. *Let γ be a section of C. Then γ is an extremal iff for every generator of automorphisms ξ on P*

$$dj^1\gamma^* i_{J^1C\xi}\Theta_\lambda = 0 \tag{4.27}$$

holds.

Proof: We have seen from the covariance identity (3.14) that $C\xi$ is the generator of invariance transformations of λ. If γ is an extremal, we get from Noether's theorem that Equation (4.27) holds for every generator of automorphisms ξ on P. Conversely, from Equation (4.25) we get for the boundary term

$$hdi_{J^1C\xi}\Theta_\lambda = (2d_i\mathcal{E}^i_j\xi^j + 2\mathcal{E}^i_j d_i\xi^j - d_i\mathcal{F}^i_J\xi^J_V - \mathcal{F}^i_J d_i\xi^J_V)\omega_0.$$

But from Equation (4.27) we see that $J^2\gamma^* hdi_{J^1C\xi}\Theta_\lambda = 0$ holds for every generator of automorphisms ξ on P. This implies that $\mathcal{E}^i_j \circ J^2\gamma = 0$, $\mathcal{F}^i_J \circ J^2\gamma = 0$, i.e. γ is an extremal.

$\diamondsuit$

Corollary 4.8 states that for this type of invariance of the gauge natural Hilbert-Yang-Mills Lagrangian, the differential conservation laws completely determine the Euler-Lagrange equations.

Chapter 5

Examples

In this chapter we shall give some examples of the previous concepts. We compute the Komar-Yang-Mills superpotential for some solutions of the Einstein-Yang-Mills equations and we comment on the conserved quantities (mass, electric charge, angular momentum). It seems that the formulae, representing the most general superpotentials, are new; we shall show that they include special cases which were described in literature. The current evaluated along an extremal can be computed directly from the principal Lepage equivalent using the explicit formula (4.14) for the lift $J^1C\xi$ or by Equation (4.26) as the exterior derivative of the Komar-Yang-Mills superpotential evaluated along a solution. This is straightforward but the result for an arbitrary generator of automorphisms of the structure bundle is quite long. In this chapter we use the terminology from the books [21, 56].

5.1 Levi-Civita-Bertotti-Robinson Solution

First we apply the result of the previous chapter to the **Levi-Civita-Bertotti-Robinson solution** of the Einstein equations. This is one of the simplest examples. We take as the structure bundle the trivial bundle $(X \times U(1), \mathrm{pr}_1, X, U(1))$ over the Levi-Civita-Bertotti-Robinson spacetime (X, g) (see e.g. [56] and for some details and interpretations [14, 39]), we suppose there exist coordinates (t, r, θ, φ) on X such that the metric g is given by

$$g = \frac{e^2}{r^2}[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)].$$

This metric g together with the $U(1)$-connection [1]

$$\Gamma = -2\,\frac{e}{r}(dt + dr)$$

is a solution of the Einstein-Maxwell equations, i.e. a solution of Equations (4.6) corresponding to the Hilbert-Yang-Mills Lagrangian λ with the only component of the Ad-invariant form h on $\mathfrak{u}(1)$ equal to 1. We denote this solution by γ_{LBR}.

We write a generator ξ of automorphisms of the structure bundle in the form

$$\xi = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial r} + \xi^3 \frac{\partial}{\partial \theta} + \xi^4 \frac{\partial}{\partial \varphi} + \zeta R_{e_1}, \tag{5.1}$$

where R_{e_1} denotes the right invariant vector field on $U(1)$ corresponding to the base vector e_1 in $\mathfrak{u}(1)$. Then we get the following coordinate expression for the pull-back $J^1\gamma^*\nu_\xi$ of the Komar-Yang-Mills superpotential ν_ξ along the solution γ_{LBR}:

$$\begin{aligned}
J^1\gamma^*_{LBR}\nu_\xi = &\left(-2\zeta e - e^2\left(\frac{2}{r}(\xi^1 + 2\,\xi^2) + \frac{\partial \xi^1}{\partial r} + \frac{\partial \xi^2}{\partial t}\right)\right)\sin\theta\, d\theta \wedge d\varphi \\
&+ \left(\frac{\partial \xi^3}{\partial t} + \frac{1}{r^2}\frac{\partial \xi^1}{\partial \theta}\right) e^2 \sin\theta\, dr \wedge d\varphi - \left(\sin\theta \frac{\partial \xi^4}{\partial t} + \frac{1}{r^2 \sin\theta}\frac{\partial \xi^1}{\partial \varphi}\right) e^2\, dr \wedge d\theta \\
&+ \left(\frac{\partial \xi^3}{\partial r} - \frac{1}{r^2}\frac{\partial \xi^2}{\partial \theta}\right) e^2 \sin\theta\, dt \wedge d\varphi + \left(\frac{1}{r^2\sin\theta}\frac{\partial \xi^2}{\partial \varphi} - \sin\theta \frac{\partial \xi^4}{\partial r}\right) e^2\, dt \wedge d\theta \\
&+ \left(2\cos\theta\,\xi^4 - \frac{1}{\sin\theta}\frac{\partial \xi^3}{\partial \varphi} + \sin\theta \frac{\partial \xi^4}{\partial \theta}\right)\frac{e^2}{r^2}\, dt \wedge dr.
\end{aligned}$$

If we choose ξ as ζR_{e_1}, we get

$$J^1\gamma^*_{LBR}\nu_{\zeta R_{e_1}} = -2\zeta e \sin\theta\, d\theta \wedge d\varphi.$$

Taking ζ as an appropriate constant and integrating on spatial spheres we obtain the electric charge e.

5.2 Reissner-Nordström Solution

Now we apply the result of the previous chapter to the **Reissner-Nordström solution** of the Einstein equations. We take as the structure bundle the trivial $U(1)$-bundle $(X \times U(1), \mathrm{pr}_1, X, U(1))$ over the Reissner-Nordström spacetime (X, g) (see e.g. [21]), we suppose there exist coordinates (t, r, θ, φ) on X such that the metric g is given by

$$g = -\left(1 - 2\,\frac{m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - 2\,\frac{m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

[1] We will not write the base vector e_1 in $\mathfrak{u}(1)$ for the $U(1)$-connections explicitly.

This metric g together with the $U(1)$-connection

$$\Gamma = -2\,\frac{e}{r}(dt + dr)$$

is a solution of the Einstein-Maxwell equations. We denote this solution by γ_{RN}.

For a generator ξ of automorphisms of the structure bundle as in (5.1) we get the following coordinate expression for the pull-back $J^1\gamma^*_{RN}\nu_\xi$ of the Komar-Yang-Mills superpotential ν_ξ along the solution γ_{RN}:

$$\begin{aligned}
J^1\gamma^*_{RN}\nu_\xi = \sin\theta\Bigg(&-2\zeta e - \frac{2}{r}\left(2\,\xi^2 e^2 + \xi^1 e^2 + \xi^1 mr\right) - \frac{\partial\xi^1}{\partial r}s - \frac{r^4}{s}\frac{\partial\xi^2}{\partial t}\Bigg) d\theta\wedge d\varphi \\
&+ \sin\theta\left(\frac{r^4}{s}\frac{\partial\xi^3}{\partial t} + \frac{\partial\xi^1}{\partial\theta}\right) dr\wedge d\varphi - \left(\frac{r^4\sin\theta}{s}\frac{\partial\xi^4}{\partial t} + \frac{1}{\sin\theta}\frac{\partial\xi^1}{\partial\varphi}\right) dr\wedge d\theta \\
&+ \sin\theta\left(\frac{2}{r}s\xi^3 + s\frac{\partial\xi^3}{\partial r} - \frac{\partial\xi^2}{\partial\theta}\right) dt\wedge d\varphi \\
&- \left(\frac{2}{r}s\sin\theta\xi^4 + s\sin\theta\frac{\partial\xi^4}{\partial r} - \frac{1}{\sin\theta}\frac{\partial\xi^2}{\partial\varphi}\right) dt\wedge d\theta \\
&+ \left(2\cos\theta\,\xi^4 - \frac{1}{\sin\theta}\frac{\partial\xi^3}{\partial\varphi} + \sin\theta\frac{\partial\xi^4}{\partial\theta}\right) dt\wedge dr, \quad s = r^2 - 2\,mr + e^2.
\end{aligned}$$

In particular, if we choose ξ as $\xi^1(\partial/\partial t)$, where ξ^1 is constant, then we have

$$J^1\gamma^*_{RN}\nu_{\xi^1(\partial/\partial t)} = -2\xi^1\left(m + \frac{e^2}{r}\right)\sin\theta\, d\theta\wedge d\varphi.$$

Thus for an appropriate choice of the constant ξ^1 we obtain, after integrating on spatial spheres at spatial infinity, the mass m.

If we choose ξ as ζR_{e_1}, we get

$$J^1\gamma^*_{RN}\nu_{\zeta R_{e_1}} = -2\zeta e\sin\theta\, d\theta\wedge d\varphi.$$

The electric charge e can be found by taking ζ as an appropriate constant and integrating over spatial spheres.

5.3 Kerr-Newman Solution

Now we consider the **Kerr-Newman solution** of the Einstein equations. We take as the structure bundle the trivial $U(1)$-bundle $(X\times U(1), \mathrm{pr}_1, X, U(1))$ over the Kerr-Newman spacetime (X, g) (see e.g. [56]), we suppose there exist coordinates (t, r, θ, φ) on X such that the metric g is given by

$$g = -\Big(1 - \frac{2mr - e^2}{u}\Big)\, dt^2 + \frac{u}{s}\, dr^2 + u\, d\theta^2 + \sin^2\theta \Big(r^2 + a^2 + \frac{a^2 \sin^2\theta}{u}(2mr - e^2)\Big)\, d\varphi^2 - \frac{a\sin^2\theta}{u}(2mr - e^2)(dt \otimes d\varphi + d\varphi \otimes dt),$$

$$s = r^2 + a^2 + e^2 - 2mr,\;\; u = r^2 + a^2\cos^2\theta.$$

This metric g together with the $U(1)$-connection

$$\Gamma = -2e\frac{r}{u}(dt - a\sin^2\theta\, d\theta)$$

is a solution of the Einstein-Maxwell equations. We denote this solution by γ_{KN}.

The Komar-Yang-Mills superpotential ν_ξ along the solution γ_{KN} for the general generator ξ of automorphisms of the structure bundle (5.1) is too long. We will consider only a few special choices of the generator. If we choose ξ as $\xi^1(\partial/\partial t)$, where ξ^1 is constant, we have

$$J^1\gamma^*_{KN}\nu_{\xi^1(\partial/\partial t)} = 2\xi^1\frac{\sin\theta}{u^3}[A(a^2 + r^2)\, d\theta \wedge d\varphi - B\sin\theta\cos\theta a^2\, dr \wedge d\varphi - Aa\, dt \wedge d\theta - Ba\cos\theta\, dt \wedge dr],$$

$$A = -mr^4 + ma^4\cos^4\theta - r^3e^2 + 3\, re^2a^2\cos^2\theta,$$

$$B = (2\, mr - e^2)a^2\cos^2\theta + 2\, mr^3 + 3\, e^2r^2.$$

Thus for an appropriate choice of the constant ξ^1 we obtain, after integrating on spatial spheres at spatial infinity, the mass m.

If we choose ξ as $\xi^4(\partial/\partial\varphi)$, where ξ^4 is constant, then we have

$$J^1\gamma^*_{KN}\nu_{\xi^4(\partial/\partial\varphi)} = 2\xi^4\frac{\sin\theta}{u^3}(-Ca\sin^2\theta\, d\theta \wedge d\varphi - Ba^3\sin^3\theta\cos\theta\, dr \wedge d\varphi + D\, dt \wedge d\theta - E\mathrm{cotg}\theta\, dt \wedge dr),$$

$$C = a^4(-mr^2 + a^2m + re^2)\cos^4\theta + a^2r(-4\, mr^3 + 5\, e^2r^2 + 3\, a^2e^2)\cos^2\theta - r^3(3\, mr^3 + a^2mr + a^2e^2),$$

$$D = a^6(m - r)\cos^6\theta + a^4(2\, re^2 + 2\, mr^2 - a^2m - 3\, r^3)\cos^4\theta - 3\, a^2r(r^4 - mr^3 + e^2r^2 + a^2e^2)\cos^2\theta + r^3(-r^4 + 2\, mr^3 - e^2r^2 + a^2mr + a^2e^2),$$

$$E = a^6\cos^6\theta + a^4(e^2 + 3\, r^2 - 2\, mr)\cos^4\theta - a^2(3\, e^2r^2 + 2\, mr^3 + a^2e^2 - 2\, a^2mr - 3\, r^4)\cos^2\theta + r^2(r^4 + 2\, a^2mr + 3\, a^2e^2).$$

Thus for an appropriate choice of the constant ξ^4 we obtain, after integrating on spatial spheres at spatial infinity, the angular momentum ma.

If we choose ξ as ζR_{e_1}, where ζ is constant, then we get

$$J^1\gamma^*_{KN}\nu_{\zeta R_{e_1}} = 2\zeta e\frac{\sin\theta}{u^2}[(-r^2 + a^2\cos^2\theta)(a^2 + r^2)\, d\theta \wedge d\varphi - a^2r\sin\theta\cos\theta\, dr \wedge d\varphi + a(-r^2 + a^2\cos^2\theta)\, dt \wedge d\theta - ra\mathrm{cotg}\theta\, dt \wedge dr].$$

Finding the electric charge e is easy now: it can be obtained by taking ζ as an appropriate constant and integrating on spatial spheres at spatial infinity.

5.4 Colored Black Hole

We take one of the simplest non-Abelian black hole solution of the Einstein-Yang-Mills equations - one of the so called **colored black holes**. We take as the structure bundle the trivial $SU(2)$-bundle $(X \times SU(2), \mathrm{pr}_1, X, SU(2))$ over the Reissner-Nordström-like spacetime (X, g) (see e.g. [58] and for some details also [5, 11]), we suppose there exist coordinates (t, r, θ, φ) on X such that the metric g is given by

$$g = -\left(1 - 2\,\frac{m}{r} + \frac{e^2+q^2}{r^2}\right) dt^2 + \left(1 - 2\,\frac{m}{r} + \frac{e^2+q^2}{r^2}\right)^{-1} dr^2 \\ + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

Let e_P for $1 \le P \le 3$ be a basis of the Lie algebra $\mathfrak{su}(2)$ given by $e_P = -\frac{i}{2}\sigma_P$ with σ_P being the Pauli matrices. Then for the structure constants we have $c^R_{PQ} = \varepsilon_{PQR}$. This metric g together with the $SU(2)$-connection

$$\Gamma = \left(-2\,\frac{e}{r}(dt + dr) + 2\,q(1-\cos\theta)\,d\varphi\right) e_3$$

is a solution of the Einstein-Yang-Mills equations, i.e. a solution of Equations (4.6) corresponding to the Hilbert-Yang-Mills Lagrangian λ with the components of the Ad-invariant form h on $\mathfrak{su}(2)$ given by $h_{PQ} = \delta_{PQ}$ (h is up to a factor the Killing form of $\mathfrak{su}(2)$). We denote this solution by γ_{CBH}.

We write a generator ξ of automorphisms of the structure bundle in the form

$$\xi = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial r} + \xi^3\frac{\partial}{\partial \theta} + \xi^4\frac{\partial}{\partial \varphi} + \zeta^P R_{e_P}, \tag{5.2}$$

where R_{e_P} denote the right invariant vector fields on $SU(2)$ corresponding to the base vectors e_P in $\mathfrak{su}(2)$. Then we get the following coordinate expression for the pull-back $J^1\gamma^*_{CBH}\nu_\xi$ of the Komar-Yang-Mills superpotential ν_ξ along the solution γ_{CBH}:

$$J^1\gamma^*\nu_\xi = \Bigg(\frac{2}{r}[\xi^1(q^2-e^2-mr) - 2\xi^2e^2] + 4\xi^4 eq(1-\cos\theta) - 2\zeta^3 e \\ - s\frac{\partial\xi^1}{\partial r} - \frac{r^4}{s}\frac{\partial\xi^2}{\partial t}\Bigg)\sin\theta\, d\theta\wedge d\varphi + \left(\frac{r^4}{s}\frac{\partial\xi^3}{\partial t} + \frac{\partial\xi^1}{\partial\theta}\right)\sin\theta\, dr\wedge d\varphi \\ - \left(\frac{r^4\sin\theta}{s}\frac{\partial\xi^4}{\partial t} + \frac{1}{\sin\theta}\frac{\partial\xi^1}{\partial\varphi}\right) dr\wedge d\theta + \left(\frac{2}{r}s\xi^3 + s\frac{\partial\xi^3}{\partial r} - \frac{\partial\xi^2}{\partial\theta}\right)\sin\theta\, dt\wedge d\varphi \\ - \left(\frac{2s}{r}\sin\theta\xi^4 + s\sin\theta\frac{\partial\xi^4}{\partial r} - \frac{1}{\sin\theta}\frac{\partial\xi^2}{\partial\varphi}\right) dt\wedge d\theta \\ + \Bigg(-\frac{4eq}{r^3}(\xi^1+\xi^2) + 2(\cos\theta - \frac{2q^2}{r^2}(1+\cos\theta))\xi^4 - 2\frac{q}{r^2}\zeta$$

$$-\frac{1}{\sin\theta}\frac{\partial\xi^3}{\partial\varphi}+\sin\theta\frac{\partial\xi^4}{\partial\theta}\Bigg)dt\wedge dr,\ \ s=r^2-2\,mr+e^2+q^2.$$

It is easy to see that, similarly as before, the mass m corresponds to $\partial/\partial t$ and the electric charge e to R_{e_3}. Moreover, if we choose ξ as $\xi^4(\partial/\partial\phi)$, where ξ^4 is constant, then after integrating on spatial spheres we see that eq must be a constant.

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