ELECTRODYNAMICS

C1 MAXWELL EQUATIONS

- THEY ARE COMPLETE SYSTEM - NO OTHER EQ.

- LINKS FIELDS $\vec{E}, \vec{B}$ TO THE CHARGES $\rho, \vec{j}$

ρ - CHARGE DENSITY ; $\vec{j}$ - FLUX

- IN AN INERTIAL FRAME, MAXWELL'S EQUATIONS ASSUME THE SHAPE

① $\operatorname{div} \vec{D} = 4\pi\rho$ GAUß-LAW

$$\int_V dV \operatorname{div} \vec{D} = \int_{\partial V} d\vec{A}\, \vec{D} = \underbrace{\psi}_{\text{EL. FLUX}} = 4\pi \int_V dV\rho = 4\pi \underbrace{Q}_{\text{TOTAL CHARGE}}$$

$$\boxed{\psi}$$

EL. CHARGES ARE SOURCES OF THE ELECTRIC FIELD, EL. FLUX ~~[illegible]~~ IS EQUAL TO THE ENCLOSED CHARGE

② $\operatorname{div} \vec{B} = 0$

$$\int_V dV \operatorname{div} \vec{B} = \int_{\partial V} d\vec{A}\, \vec{B} = \underbrace{\phi}_{\text{MAG. FLUX}} = 0$$

(IN CLOSE SURFACE)

THERE ARE NO MAGNETIC CHARGES, MAGNETIC FLUX THROUGH A CLOSED SURFACE IS ALWAYS ZERO.

③ $\operatorname{rot} \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B}$ FARADAYS LAW OF INDUCTION

$$\int_A d\vec{A} \cdot \operatorname{rot} \vec{E} = \underbrace{\int_{\partial A} d\vec{S}\, \vec{E}}_{\sim \text{INDUCED VOLTAGE ALONG LOOP}} = -\frac{1}{c} \frac{d}{dt} \int_A d\vec{A} \cdot \vec{B} = -\frac{1}{c} \frac{d}{dt} \underbrace{\phi}_{\text{MAG. FLUX DEFINED OVER SOME SURFACE}}$$

IN THE MAGNETIC FLUX THROUGH A LOOP CHANGES, THERE IS AN INDUCED ELECTRIC FIELD.

MINUS SIGN FROM LENZ RULE.

④ $\text{rot}\,\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\vec{D}$ AMPÈRS LAW

$$\int_A d\vec{A}\,\text{rot}\,\vec{H} = \int_{\partial A} d\vec{s}\,\vec{H} = \frac{4\pi}{c}\underbrace{\int d\vec{A}\cdot\vec{j}}_{} + \frac{1}{c}\frac{d}{dt}\underbrace{\int_A d\vec{A}\cdot\vec{D}}_{}$$

$$= \frac{4\pi}{c}\cdot I \quad + \frac{1}{c}\frac{d}{dt}\psi$$

THERE IS A MAGNETIC FIELD AROUND A CURRENT OR AROUND CHANGING ELECTRIC FLUX.

C2 ELECTRODYNAMICS IN VACUUM

- THE FIELDS $\vec{D}$ AND $\vec{H}$ ARE RELATED TO $\vec{E}$ AND $\vec{B}$ BY THE DIELECTRIC CONSTANT ϵ AND THE PERMEABILITY μ

$\vec{D} = \epsilon\cdot\vec{E}$ — DIELECTRIC CONSTANT

$\vec{B} = \mu\cdot\vec{H}$ — PERMEABILITY

$\epsilon = \mu = 1$ IN VACUUM

C3 CHARGE CONSERVATION

- EL. CHARGE IS CONSERVED, SO THE DYNAMICS OF FIELD MUST RESPECT THAT!

$$\text{rot}\,\vec{H} = \frac{4\pi}{c}\left(\vec{j} + \frac{1}{4\pi}\frac{d}{dt}\vec{D}\right) = \frac{4\pi}{c}\cdot\vec{z}$$

$\vec{z}$ ~ DISPLACEMENT CURENT

$$\text{div}\,\vec{z} = \text{div}\,\vec{j} + \frac{1}{4\pi}\frac{d}{dt}\underbrace{\text{div}\,\vec{D}}_{\text{the } 4\pi\rho} = \text{div}\,\vec{j} + \frac{\partial}{\partial t}\rho = 0$$

CONSERV. LAW IS ALREADY IN MAXWELLS EQUATIONS

BECAUSE DIV ROT $\vec{H} = 0$

C4 OHMS LAW - FENOMENOLOGICAL RELATION

$\vec{j} = \sigma\cdot\vec{E}$, WITH CONDUCTIVITY (TENSOR) σ

$j_i = \sigma_{ij}\cdot E_j$

[USUAL FOR $\underbrace{j\cdot A}_{I} = \sigma\cdot A\cdot\frac{z}{z}\,E = \underbrace{\frac{\sigma\cdot A}{z}}_{\frac{1}{R}}\cdot\frac{E\cdot z}{}$]

U

C5 LORENTZ FORCE

$$\vec{f} = \rho \cdot \vec{E} + \frac{1}{c} \vec{j} \times \vec{B}$$ (CAREFUL, RELATIVISTIC FORCE)

- IT'S ONLY VALID IN A SPECIFIC INERTIAL FRAME AND ONE NEEDS A RELATIVISTIC EQUATION OF MOTION, OTHERWISE $\beta \ll 1$, $|v| \ll c$

C6 ENERGY CONSERVATION IN ELECTRODYNAMICS → POYNTING LAW

(ASSUME VACUUM $\epsilon = \mu = 1 \rightarrow \vec{E} = \vec{D}$ AND $\vec{B} = \vec{H}$

TAKE THE TWO ROT-EQUATIONS AND ASSEMBLE

$$\vec{E} \cdot \mathrm{rot}\vec{B} - \vec{B} \cdot \mathrm{rot}\vec{E} = \frac{4\pi}{c} \vec{j} \cdot \vec{E} + \frac{1}{c}\left(\vec{E} \cdot \frac{\partial}{\partial t} \vec{E} + \vec{B} \frac{\partial}{\partial t} \vec{B}\right) = -\mathrm{div}(\vec{E} \times \vec{B})$$

SIGN SWITCH! LENZ-RULE

ENERGY DENSITIES OF ELECTRIC PART OF THE FIELD:

$$w_{el} = \frac{\vec{E}^2}{8\pi} \rightarrow \frac{\partial}{\partial t} w_{el} = \frac{1}{4\pi} \vec{E} \frac{\partial}{\partial t} \vec{E}$$

MAGNETIC PART OF THE FIELD

$$w_{mag} = \frac{\vec{B}^2}{8\pi} \longrightarrow \frac{d}{dt} w_{mag} = \frac{1}{4\pi} \vec{B} \frac{\partial}{\partial t} \vec{B}$$

BECAUSE $w_{el} = -\frac{1}{2} \int_V dV \rho \phi = -\frac{1}{8\pi} \int_V dV \, \Delta\phi \cdot \phi =$

$$= \frac{1}{8\pi} \int_V dV (\nabla \phi)^2$$

$$\longrightarrow \frac{c}{4\pi} \mathrm{div}(\vec{E} \times \vec{B}) + \vec{j} \cdot \vec{E} = -\frac{\partial}{\partial t}(w_{el} + w_{mag}) =$$

$$= \mathrm{div} \vec{S} + j \cdot \vec{E}$$

ENERGY CONSERVATION IN THE FORM OF CONTINUITY RELATION.

$\vec{S}$ - POYNTING VECTOR (DESCRIBES ENERGY TRANSPORT OF ENERGY SPECTRUM)

ENERGY CONSERVATION: CHANGE IN ENERGY IS EQUAL TO THE ENERGY FLUX ACROSS THE BOUNDARY AND TO THE DISSIPATION.

C7 MOMENTUM CONSERVATION

LORENTZ FORCE: $\vec{F} = \int_V dV \underbrace{\vec{f}}_{\text{LORENTZ FORCE DENSITY}} = \int_V dV \left(\rho \cdot \vec{E} + \frac{1}{c} \vec{j} \times \vec{B}\right) = \frac{d}{dt}\vec{P}$ EQUATION OF MOTION

FORCES ACTING ON A CHARGE AND ACCELERATE IT.

EXPRESS ρ AND $\vec{j}$ WITH $\vec{E}$ AND $\vec{B}$

$\operatorname{div}\vec{E} = 4\pi\rho \longrightarrow \rho = \frac{1}{4\pi}\operatorname{div}\vec{E}$

$\operatorname{rot}\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E} \longrightarrow \vec{j} = \frac{c}{4\pi}\left(\operatorname{rot}\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E}\right)$

$\rightarrow \frac{d}{dt}\vec{P} = \frac{1}{4\pi}\int_V dV \left(\vec{E}\cdot\operatorname{div}\vec{E} + \operatorname{rot}\vec{B}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\times\vec{B}\right) =$

$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial B}{\partial t}$

$-\frac{1}{c}\frac{\partial}{\partial t}\vec{B} = \operatorname{rot}\vec{E}$

$\rightarrow \frac{d}{dt}\vec{P} = \frac{1}{4\pi}\int_V dV \Big(\vec{E}\cdot\operatorname{div}\vec{E} + \underbrace{\vec{B}\cdot\operatorname{div}\vec{B}}_{0} + \operatorname{rot}\vec{B}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\cdot$

$\cdot(\vec{E}\times\vec{B}) - \vec{E}\times\operatorname{rot}\vec{E}\Big)$

CAN BE ADDED BECAUSE ITS ZERO

$\boxed{\cancel{\frac{d}{dt}\vec{P} = 0}}$? MAYBE?

$\rightarrow \frac{d}{dt}\Big(\vec{P} + \frac{1}{c^2}\int_V dV \underbrace{\vec{S}}_{\text{POYNTING VECTOR}}\Big) = \frac{1}{4\pi}\int_V dV \left(\vec{E}\cdot\operatorname{div}\vec{E} + \vec{B}\cdot\operatorname{div}\vec{B} + \operatorname{rot}\vec{B}\times\vec{B} + \operatorname{rot}\vec{E}\times\vec{E}\right)$

RATE OF CHANGE OF MOMENTUM CARRIED BY THE FIELD

$\rightarrow$ SOURCE IF (BRACKETS) CAN BE WRITTEN AS A DIVERGENCE.

$(\vec{E}\cdot\operatorname{div}\vec{E})_i = E_i\,\partial_j E_j$ (VECTORIAL PROPERTY CARRIED BY E_i)

$(\vec{E}\times\operatorname{rot}\vec{E})_i = \epsilon_{ijk}\cdot E_j\cdot\epsilon_{klm}\,\partial_l E_m = (-\delta_{im}\delta_{jl} + \delta_{il}\delta_{jm})\cdot$

$\cdot E_j\,\partial_l E_m = E_j\,\partial_j E_i - E_j\,\partial_i E_j$

$$\to (\vec{E} \cdot \operatorname{div} \vec{E} + \operatorname{rot} \vec{E} \times \vec{E})_i = E_i \partial_j E_j + E_j \partial_j E_i - E_j \partial_i E_j =$$

$$= \partial_j (E_i E_j) - \frac{1}{2} \partial_i (E_j E_i) = \partial_j \left(E_i E_j - \frac{1}{2} \delta_{ij} (E_k E_k)\right)$$

SAME ARGUMENT FOR $\vec{B} \cdot \operatorname{div} \vec{B} + \operatorname{rot} \vec{B} \times \vec{B}$

$$\to \frac{d}{dt}\left(\vec{p} + \frac{1}{c^2} \int_V dV \vec{S}\right) = \frac{1}{4\pi} \int_V dV \partial_j \left(E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} \cdot (E_k E_k + B_k B_k)\right)$$

ENERGY DENSITIES: $w_{el} = \frac{1}{8\pi} \vec{E}^2$ AND $w_{mag} = \frac{1}{8\pi} \vec{B}^2$

$\to W = \vec{E}^2 + \vec{B}^2 = 8\pi$ (TOTAL ENERGY DENSITY)

MAXWELL STRESS-TENSOR

$$T_{ij} = \frac{1}{4\pi}\left(E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} W\right)$$

TRACE $\quad \operatorname{tr} T = \sum_i T_{ii} = \frac{1}{4\pi}\left(\vec{E}^2 + \vec{B}^2 - \frac{1}{2}(\vec{E}^2 + \vec{B}^2)\right) = w_{el} + w_{mag} =$

$$= \underline{\frac{W}{8\pi}}$$

MOMENTUM CONSERVATION: COLLECT ALL RESULTS

$$\to \frac{d}{dt}\left(\vec{p} + \frac{1}{c^2} \int_V dV \vec{S}\right)_i = \int_V dV \partial_j T_{ij} = \oint_{\partial V} dA_j T_{ij}$$

T_{ij} ~ STRESS SURFACE-NORMALISED FORCE INTO i-DIRECTION PULLING ON THE SURFACE WITH NORMAL INTO j-DIRECTION CORESPONDS TO STRESS TENSOR $\underline{\sigma}$ (REYNOLDS IN FLUID MECHANICS).

C8 ELECTROMAGNETIC WAVES IN VACCUM

- LETS INTRODUCE VECTOR POTENTIAL $\vec{A}$ AS $\vec{B} = \text{rot}\,\vec{A}$

$$\text{rot}\,\vec{E} = -\underbrace{\partial_{ct}}_{\partial_{ct} = \frac{1}{c}\frac{\partial}{\partial t}}\vec{B} \rightarrow \text{rot}\,(E + \partial_{ct}\vec{A}) = 0$$

- IMPLYING THAT THERE MUST BE A SCALAR POTENTIAL $\vec{E} + \partial_{ct}\vec{A} = -\nabla\phi$ BECAUSE $\text{rot}\,\nabla\phi = 0$

CONVENTION

$$\vec{E} = -\nabla\phi - \partial_{ct}\vec{A} \rightarrow \text{div}\,\vec{E} = -\Delta\phi - \partial_{ct}\,\text{div}\,\vec{A} = 4\pi\rho$$

$$\vec{B} = \text{rot}\,\vec{A} \longrightarrow \text{rot}\,\text{rot}\,\vec{A} = \frac{4\pi}{c}\vec{j} + \partial_{ct}(-\nabla\phi - \partial_{ct}A) =$$

$$= \nabla\,\text{div}\,\vec{A} - \Delta\vec{A}$$

$\text{rot}\,\text{rot}\,\vec{A} = \nabla\,\text{div}\,\vec{A} - \Delta\vec{A}$; TWO COUPLED EQUATIONS

$$\Delta\phi + \partial_{ct}\,\text{div}\,\vec{A} = -4\pi\rho$$
$$\Delta\vec{A} - \partial_{ct}^2\vec{A} = \frac{4\pi}{c}\vec{j} + \nabla(\text{div}\,\vec{A} + \partial_{ct}\phi)$$

C9 GAUGE TRANSFORMATION

$\vec{B}$ (THE PHYSICAL FIELD [MAGNETIC]) IS NOT CHANGED IF $\nabla\chi$ IS ADDED TO THE POTENTIAL $\vec{A}$

$$\text{rot}\,\nabla\chi = 0$$

$\vec{E}$ CAN'T CHANGE EITHER: ONE MUST CHANGE ϕ BY $-\partial_{ct}\chi$ (OR IN OTHER WORDS, SUBTRACT IT FROM ϕ

GAUGE TRANSFORMATION

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \nabla \chi$$
$$\phi \longrightarrow \phi' = \phi - \partial_{ct} \chi$$

THE GAUGE TRANSFORMATION LEAVE THE FIELDS INVARIANT, BUT CAN BE USED TO DECOUPLE THE RELATIONS FOR THE POTENTIALS (FIELD EQUATIONS).
A SPECIFIC GAUGE TRANSFORMATION CAN BE CHOSEN TO FULFILL A GAUGE FIXING CONDITION.
LORENZ - GAUGE $\partial_{ct}\phi + \operatorname{div}\vec{A} = 0$

$$\Delta\phi - \partial_{ct}^2\phi = -4\pi\rho = \Box\phi$$
$$\Delta\vec{A} - \partial_{ct}^2\vec{A} = \frac{4\pi}{c}\vec{j} = \Box\vec{A}$$

~ WAVE EQUATIONS WITH SOURCES ρ AND $\vec{j}$.

D'ALEMBERT OPERATOR:
$$\Box = \partial^\mu\partial_\mu = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \frac{\partial^2}{\partial_x^2} - \frac{\partial^2}{\partial_y^2} - \frac{\partial^2}{\partial_z^2}$$
LAPLACE OPERATOR IN MINKOWSKI SPACE = D'ALEMBERT OPERATOR

WHAT HAPPENS TO THE GAUGE FIELD χ?
$\partial_{ct}(\phi - \partial_{ct}\chi) + \operatorname{div}(\vec{A} + \nabla\chi) = 0$ IN LORENZ GAUGE
IT IS CONSISTENT IF $\Delta\chi - \partial_{ct}^2\chi = 0$
$\longrightarrow$ POTENTIALS ϕ AND $\vec{A}$ ARE LINKED TO THE SOURCES $\vec{j}$ AND ρ WITH A WAVE EQUATION. CHANGES IN $\vec{j}$ OR ρ PROPAGATION AT THE SPEED C. (LIGHT SPEED ZVEJKU).

C10 COUPLING THE POTENTIALS TO THE CHARGES

($\Box$ REPLACES Δ, AND WE NEED A NEW TIME-DEPENDENT GREEN FUNCTION)

TYPE OF EQUATION $\Delta\psi - \partial_{ct}^2\psi = -4\pi f$

CONSTRUCT A GREEN-FUNCTION FOR DUALING WITH THE CHARGE/CURRENT DISTRIBUTION. ~~AND WITH~~

GREEN-FUNCTION G, DEFINED THROUGH

$$(\Delta - \partial^2_{ct})\, G(\vec{r};t;\vec{r}';t') = -4\pi\, \delta_D(\vec{r}-\vec{r}')\, \delta_D(t-t')$$

THE FIELD OR ANY CHARGE DISTRIBUTION IS CONSTRUCTED BY SUPERPOSITION, BECAUSE MAXWELL-ELECTRODYNAMICS IS LINEAR.

$$\psi(\vec{r};t) = \int d^3r' \int dt'\; G(\vec{r};t;\vec{r}';t') \cdot f(\vec{r}';t')$$

~ CONVOLUTION RELATION

C11 LIENARD-WIECHERT POTENTIALS

→ CONSTRUCT THE GREEN FUNCTION FOR LINEAR POTENTIAL PROBLEMS.

FOURIER SPACE - TRANSFORMATION THE DIFFERENTIAL EQUAT. INTO ALGEBRAIC (INVERSION IS EAZY)

$$G(\vec{r};t;\vec{r}';t') = \int d^3k \int d\omega\; G(\vec{k};\omega) \cdot \exp\left[+i\vec{k}(\vec{r}-\vec{r}')\right] \cdot \exp\left[-i\omega(t-t')\right]$$

MINUS CONVENTION

$$\Box\, G(\vec{r};t;\vec{r}';t') = \int d^3k \int d\omega\; G(\vec{k};\omega)\, \Box \exp\left[ik(\vec{r}-\vec{r}')\right] \cdot$$

WITH $\Box = \Delta - \partial^2_{ct}$

D'ALEMBERT OPERATOR

$$\cdot \exp\left[-i\omega(t-t')\right] = \int d^3k \int d\omega\; G(\vec{k};\omega) \cdot$$

$$\cdot \left[k^2 - \left(\frac{\omega}{c}\right)^2\right] \exp\left[ik(\vec{r}-\vec{r}')\right] \cdot \exp\left[-i\omega(t-t')\right]$$

MEZI KROK

$$= \int d^3k \int d\omega\; G(\vec{k};\omega)\left(\Delta \exp(ik(\vec{r}-\vec{r}')) - \partial^2_{ct} \exp(-i\omega(t-t'))\right)$$

LET'S COMPARE IT WITH THE RIGHT SIDE OF THE WAVE EQ.:

$$= \int \frac{d^3k}{(2\pi)^3} \cdot \int \frac{d\omega}{2\pi} \cdot (4\pi) \cdot \exp(i\vec{k}(\vec{r}-\vec{r}\,')) \cdot \exp(-i\omega(t-t'))$$

$$\rightarrow \boxed{G(\vec{k};\omega) = \frac{1}{4\pi^3} \cdot \frac{1}{k^2 - \frac{\omega^2}{c^2}}}$$

GREEN FUNCTION IN FOURIER SPACE

TRANSFORM BACK TO REAL SPACE:

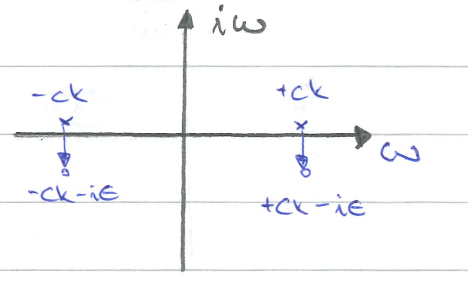
$$G(\vec{r}-\vec{r}\,';t-t') = \frac{1}{4\pi^3} \cdot \int d^3k \int d\omega \underbrace{\frac{c^2}{(ck)^2-\omega^2}}_{=G(\vec{k};\omega)} \exp\cdot[i\vec{k}(\vec{r}-\vec{r}\,')] \cdot \exp[-i\omega(t-t')]$$

BUT THE INTEGRAND IS NOT DEFINIED AT $\omega = \pm ck$

$\rightarrow$ USE TOOLS FROM COMPLEX-CALCULUS FOR PERFORMING THE INTEGRATION AT $\omega = \pm ck$

$d\omega$ - INTEGRATION MOVE TO THE COMPLEX PLANE

LET'S OFFSET THE POLES OF THE INTEGRAND BY A SMALL AMOUNT $\underline{\epsilon}$.



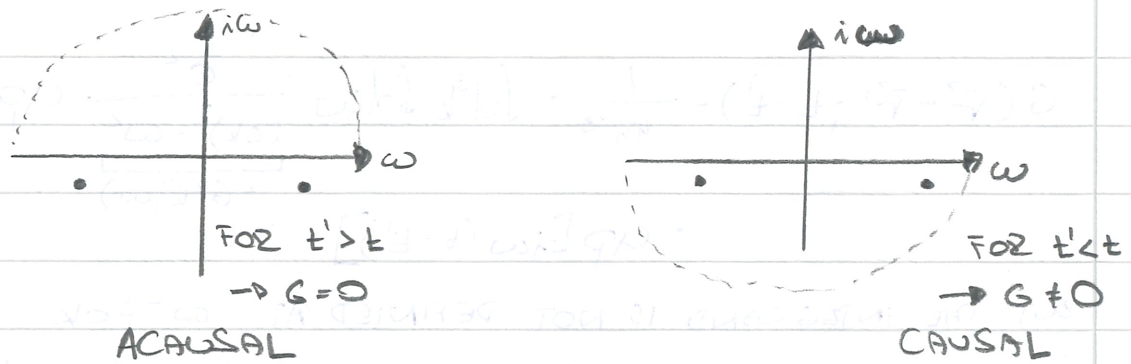
FOR $d\omega \quad \dfrac{\exp(-i\omega t)}{(ck)^2-(\omega-i\epsilon)^2}$

AS PART OF THE GREEN'S FUNCTION

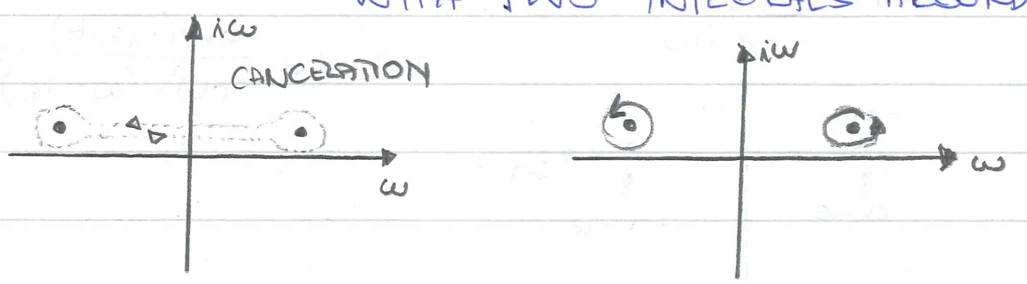
$$G(\vec{\rho};\tau) = \frac{1}{4\pi^3} \int d^3k \int d\omega \frac{\exp[i(\vec{k}\vec{\rho}-\omega t)]}{k^2 - \left(\frac{\omega - i\epsilon}{c}\right)^2}$$

FOR APPLYING THE RESIDUE THEOREM ONE NEEDS A CLOSED LOOP $\longrightarrow$ INTEGRAND SHOULD DROP AT LEAST LIKE $1/\omega^2$, OTHERWISE THE CLOUSING ...

IF THE LOOPS DOES NOT CONTAIN ANY OF THE POLES, THE RESULT = 0 DUE TO HOLOMORPHY (COMPLEX DIFFERENTIABILITY).
WE CAN CHOOSE THE CLOSING ARC TO CONTAIN THE POLES FOR $t > t'$; $\tau > 0$ (CAUSAL SOLUTION) OR NOT TO CONTAIN THE POLES FOR $t < t'$; $\tau < 0$ (ACAUSAL SOLUTION)



FOR A HOLOMORPHIC FUNCTION THE INTEGRATION PATH DOES NOT MATTER:
→ ~~MORPHOLOGICAL~~ INTEGRATION PATH TO → WE ARE LEFT WITH TWO INTEGRALS AROUND TWO POLES



EFFECTIVELY, THERE ARE TWO INDIVIDUAL LOOPS LEFT WITH INDIVIDUAL SENSES OF ROTATION → IDENTICAL RESIDUES, DUE TO SYMMETRY.

C13 COVARIANT FORMULATION OF ELECTRODYNAMICS

LORENTZ - FORCE + FIELD TENSOR (FARADAY TENSOR)

$$\frac{d\vec{p}}{dt} = q \cdot \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

$\frac{d\vec{p}}{dt}$ → MOTION OF A CHARGE

6.

ONE OBSERVER SEES ELECTRIC & MAGNETIC FIELD
A OBSERVES A CHARGE, IN THAT FRAME THE
MOTION OF A CHARGE IS GIVEN BY $\frac{d\vec{p}}{dt}$,
ANOTHER OBSERVER MIGHT SEES DIFFERENT $\vec{E}$ & $\vec{B}$
BUT FOR THAT OBSERVER THE TRAJECTORY IS
OF COURCE DIFFERENT. THE $\vec{E}$ & $\vec{B}$ CHANGE
CONSISTENTLY AS WELL AS THE TRAJECTORY OF
THAT PARTICULAR PARTICLE (LORENTZ TRANSFORM).

LET'S TRANSFORM ~~IT~~ INTO A COVARIANT EQUATION,
WE CAN DO TWO THINGS - FIRST MULTIPLY WITH
$\gamma \frac{\vec{v}}{c}$; AND MULTIPLYING WITH γ

COMBINE
$$\left\{ \begin{array}{ll} m\gamma \frac{d}{dt}(\gamma c) = \gamma \frac{q}{c} \vec{E}\cdot\vec{v} & / \gamma \frac{\vec{v}}{c} \\ m\gamma \frac{d}{dt}(\gamma \vec{v}) = \gamma q \cdot (\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}) & / \gamma \end{array} \right.$$

$$m \frac{d}{d\tau} u^\mu = q \cdot \overbrace{\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_x & B_y \\ E_y & B_x & 0 & -B_z \\ E_z & -B_y & B_z & 0 \end{pmatrix}}^{F_{\mu\nu}} \cdot u_\nu$$

WITH FOUR VELOCITY $u^\mu = \gamma(c; \vec{v}) = (\gamma c; \gamma \vec{v})$ AND
PROPER TIME $\frac{dt}{d\tau} = \gamma$; WHERE $F_{\mu\nu}$ IS ANTISYMETRIC
TENSOR CONSERVE NORMALISATION OF u^μ (FOUR VECTOR OF A VELOCITY)
IT TRACE IS <u>ZERO</u>.
FOUR VELOCITY $u_\mu u^\mu = c^2$, VELOCITIES IN MINKOWSKI
SPACE DON'T FORM VECTOR SPACE, WE CANNOT SIMPLY ADD
FOUR VELOCITIES. IF WE WANT TO ADD VELOCITIES WE
HAVE TO CONSERVE THIS NORMALIZATION $u_\mu u^\mu = c^2$,
IF FORCE LIKE "LORENTZ FORCE" ACTS ON PARTICLE.

$$\frac{1}{2}\frac{d}{d\tau}(u_\mu u^\mu) = 0 = u_\mu \frac{du^\mu}{d\tau} = u_\mu \cdot \frac{q}{Mc} F^{\mu\nu} u_\nu =$$

$$= \frac{q}{Mc} \underbrace{F^{\mu\nu}}_{\text{ANTISYMETRIC}} \underbrace{u_\mu u_\nu}_{\text{SYMETRIC}} = 0$$

ANTISYMETRIC - THIS TENSOR COMES IN IN FIELD EQUATION

- FIELD EQUATION AND CHARGE CONSERVATION.

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} \partial_\nu j^\nu = 0 \quad \xrightarrow{/\cdot\partial_\nu}$$

MAXWELL FIELD EQUATION ~~CONSERVATION OF CHARGE~~

STRONG CONSTRAIN HOW PARTICLES SHOULD MOVE, SIMILAR CONSTREIN HOW THE FIELDS GENERATED BY THE PARTICLES.

$$\partial_\nu \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} \cdot \underbrace{\partial_\nu j^\nu = 0}_{\text{CONSERVATION OF CHARGE}}$$

$\partial_\nu \partial_\mu$ - SYMETRIC TENSOR

$F^{\mu\nu}$ - ANTISYMETRIC TENSOR - HIDDEN CONSERV. OF CHARGE.

CONSTRUCTION OF ~~A~~ THE FIELD EQUATION,

- LINEAR
- FIRST ORDER IN $\underline{F}$, SECOND ORDER IN $\underline{A}$

VECTOR FIELD THAT IS LORENTZ VECTOR

POSTULATE A 4-POTENTIAL A_μ AS A LORENTZ-VECTOR TO CONTAIN ϕ AND $\vec{A}$

$$A_\mu \longrightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$ IS AUTOMATICALLY ANTISYMETRIC

$$\partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) =$$

$$= \underbrace{\partial^\mu \partial_\mu}_{=\Box \text{ D'ALEMB. OP.}} A_\nu - \partial_\nu \underbrace{\partial^\mu A_\mu}_{=0 \text{ IN LORENZ GAUGE TRANSF}} = \frac{4\pi}{c} j_\nu$$

GAUGE TRANSFORMATION

GAUGE FUNCTION

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$$

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} = \partial_\mu (A_\nu + \partial_\nu \chi) - \partial_\nu (A_\mu + \partial_\mu \chi) =$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{\partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi}_{0} = F_{\mu\nu}$$

ANTISYMMETRY OF F AND THE DUAL TENSOR $\tilde{F}$

$F_{\mu\nu}$ HAS 6 ENTRIES IN 4 DIMENSIONS DUE TO ANTISYMMETRY $\leftarrow 3\times \vec{E}$ & $3\times \vec{B}$.

$$F_{\mu\nu} = -F_{\nu\mu} \longrightarrow \underbrace{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}}_{\text{CYCLIC PERMUTATION}} = 0 \overset{*}{=} \text{ BIANCHI-IDENTITY}$$

$$\overset{*}{=} \partial_\lambda (\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu (\partial_\nu A_\lambda - \partial_\lambda A_\nu) + \partial_\nu (\partial_\lambda A_\mu - \partial_\mu A_\lambda) = 0$$

(TERMS CANCEL PAIRWISE)

THE BIANCHI IDENTITY IS THE FIELD EQUATION FOR VACUUM SOLUTIONS AND PROVIDES MEANS OF PROPAGATION OF THE FIELD:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\lambda\nu} + \partial_\nu F_{\lambda\mu} = 0 \quad / \cdot \partial^\lambda$$

$$\underbrace{\partial^\lambda \partial_\lambda}_{\square} F_{\mu\nu} + \underbrace{\partial^\lambda \partial_\mu F_{\lambda\nu}}_{0} + \underbrace{\partial^\lambda \partial_\nu F_{\lambda\mu}}_{0 \text{ IN VACUUM}} = 0 \longrightarrow \square F_{\mu\nu} = 0$$

WITH $\square = \partial_\mu \partial^\mu$ WAVE EQUATION

IF ONE HAS A STATIC SITUATION AND CHOOSES THE CORRECT FRAME IN WHICH THE CHARGE IS AT REST.

$\Delta F_{\mu\nu}=0 \rightarrow \Delta\vec{E}=0=\Delta(\nabla\phi)=\nabla(\Delta\phi)\longrightarrow\Delta\phi=0$

WITH THE CORECT BOUNDARY CONDITION AT INFINITY.

FOR STATIC CASS $\Delta F_{\mu\nu}=0$.

FIELD CAN ACTUALLY BE IN A REGION OF SPACE WHERE THERE ARE NO CHARGES AROUND $\rightarrow$ BIANCHI IDENT

FIELD DUAL: WRITE THE BIANCHI IDENTITY LIKE A FIELD EQUATION

$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0$

BECAUSE OF THE CYCLIC PROPERTY

$\epsilon^{i\lambda\mu\nu}\partial_\lambda F_{\mu\nu}=\partial_\lambda\cdot\underbrace{(\epsilon^{i\lambda\mu\nu}F_{\mu\nu})}_{2\cdot\tilde{F}^{i\lambda}}=2\cdot\partial_\lambda\tilde{F}^{i\lambda}=0$

TAKES THE SHAPE OF FIELD EQUATION

ELECTRODYNAMICS: ① $\partial_\mu F^{\mu\nu}=\frac{4\pi}{c}j^\nu$ (FIELD EQUATION)

NATURE DECIDES TO HAVE ONLY ELECTRON CHARGE $\rightarrow$ NO MONOPOLES. $\longrightarrow$ ② $\partial_\mu\tilde{F}^{\mu\nu}=0$ (BIANCHI IDENTITY)

③ $\partial_\mu j^\mu=0$

④ A^μ AND j^μ ARE LORENTZ VECTORS

IN VACUUM: $\partial_\mu F^{\mu\nu}=\partial_\mu\tilde{F}^{\mu\nu}=0$

$\vec{E}\rightarrow\vec{B}$; $\vec{B}\rightarrow-\vec{E}$ ELECTROMAGNETIC DUALITY

- IN VACUUM BOTH WOULD BE IDENTICAL, ELECTROMAGNETICS IN VACUUM IS IDENTICAL IF $\vec{E}\rightarrow\vec{B}$; $\vec{B}\rightarrow-\vec{E}$ "ELECTROMAGNETIC DUALITY"

INVARIANTS (OF FIELDS) UNDER LORENTZ TRANSFORMATION

$F_{\mu\nu}F^{\mu\nu}=2\cdot(\vec{E}^2-\vec{B}^2)=\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ (SCALAR)

$F_{\mu\nu}\tilde{F}^{\mu\nu}=4\cdot\vec{E}\cdot\vec{B}$ (PSEUDO SCALAR)

ELECTRODYNAMICS FROM A VARIATIONAL PRINCIPLE

- LAGRANGE DENSITY – LORENTZ SCALARS → MAKES SURE THAT FIELD EQUATION IS COVARIANT. LAGRANGE FUNCTION NOT A STATEMENT ABOUT ENERGY BUT RATHER ABOUT CAUSALITY
- SQUARES OF FIRST DERIVATES (OF A_μ) → SECOND ORDER FIELD EQUA.
- COUPLING INVOLVING THE POTENTIAL → LINEAR FIELD EQ.

LAGRANGE DENSITY $\mathcal{L}$

$$\mathcal{L} = \frac{1}{16} \underbrace{F_{\mu\nu} F^{\mu\nu}}_{\text{FIRS DERIVATIVE OF A}} - \frac{1}{c} j_\mu A^\mu \quad \rightarrow \quad S = \int d^4x\, \mathcal{L}$$

HAMILTONS PRINCIPLE $\delta S = 0 \longrightarrow \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$

$$\delta S = \delta \int d^4x \, F_{\mu\nu} F^{\mu\nu} = - \int d^4x A_\mu \Box A^\mu \; ; \; \delta S = 0 \rightarrow \Box A_\mu = \frac{4\pi}{c} j_\mu$$

HAMILTONIAN DENSITY $\mathcal{H}$ $\quad \frac{1}{4} F_\mu F^\mu$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_t A_\mu)} \cdot \partial_t A_\mu - \mathcal{L} = \dots = \frac{1}{2} \cdot (\vec{E}^2 + \vec{B}^2) = F_{\mu t} \partial_t A_\mu + \frac{1}{4} F_\mu F^\mu$$

$F_{\mu t}$

ENERGY DENSITY OF THE FIELD, NOT A $\underbrace{\text{LORENTZ - INVARIANT}}_{*}$

* WHEN WE DO LORENTZ TRANSFORMATION THIS IS NOT CONSERVED THEREFORE IT IS LORENTZ* - INVARIANT.

PROPAGATION OF PHOTONS

- PHOTONS ARE A WAY OF PROPAGATION OF A FIELD ITSELF AWAY FROM A CHARGE, WHAT WE KNOW ABOUT PHOTONS SHOULD FOLLOW FROM BIANCHI IDENTITY.

$\Box F_{\mu\nu} = 0$, SOLVE WITH $F_{\mu\nu} = Q_{\mu\nu} \cdot \exp(i \overbrace{k_\sigma x^\sigma}^{(\omega t - \vec{k}\vec{x})})$

$$\Box F_{\mu\nu} = Q_{\mu\nu} \Box \exp \cdot (i k_\sigma x^\sigma) = Q_{\mu\nu} \cdot \exp(i k_\sigma x^\sigma) \cdot \underbrace{k_\sigma k^\sigma}_{0} = 0$$

WAVE VECTOR IS A MULTIVECTOR

$$k_\nu k^\nu = \left(\frac{\omega}{c}\right)^2 - \vec{k}^2 = 0 \quad \rightarrow \quad \omega = \pm ck$$

NO DISPERSION IN VACUUM

GROUP VELOCITY $v_g = \frac{\partial \omega}{\partial k} = c = \frac{\omega}{k} = v_{ph}$ = PHASE VELOCITY

PHOTON TRAJECTORY $x^\nu(\lambda)$

$$k^\nu = \frac{dx^\nu}{d\lambda}$$ - TANGENT TO THE TRAJECTORY

$$x^\nu x_\nu = \frac{dx^\nu}{d\lambda} \cdot \frac{dx_\nu}{d\lambda} (d\lambda)^2 = \underbrace{k_\nu k^\nu}_{0} (d\lambda)^2 = 0$$

$\rightarrow x = \pm ct$ FROM $x_c - x^\nu = 0$

C14 THE PLANCK SPECTRUM

- PHOTON ENERGY DISTRIBUTION FROM AN IDEALLY ABSORBING/EMITTING BODY IN THERMAL EQUILIBRIUM

HISTORICALLY: SECOND QUANTUM MECHANICAL SYSTEM AFTER THE QUANTUM MECHANICAL KEPLER-PROBLEM - HYDROGEN

- STATISTICAL MECHANICS

$$p(\epsilon, T) \sim \exp\left(-\frac{\Delta\epsilon}{kT}\right)$$ BOLTZMANN-FACTOR

PROBABILITY OF A THERMAL FLUCTUATION $\Delta\epsilon$ AT TEMPERATURE T

$kT = \frac{1}{40}$ eV AT $T = 300$K $\rightarrow$ THERMAL FLUCTUATIONS ARE RELEVANT FOR ATOMS

$k = 10^{-23}$ J/K

WHY THIS SHAPE OF THE BOLTZMANN FACTOR?

→ TRANSITIVITY

RATIO $\frac{p(\epsilon_2;T)}{p(\epsilon_1;T)}$ CAN ONLY BE A FUNCTION OF $\epsilon_2-\epsilon_1$

$$g(\epsilon_3-\epsilon_1)=\frac{p(\epsilon_3;T)}{p(\epsilon_1;T)}=\frac{p(\epsilon_3;T)}{p(\epsilon_2;T)}\cdot\frac{p(\epsilon_2;T)}{p(\epsilon_1;T)}=g(\epsilon_3-\epsilon_2)\,g(\epsilon_2-\epsilon_1)$$

UNIQUELY SOLVED BY $g(\epsilon)=\exp(-\beta;\epsilon)$

(LOOK AT THE LOGARITHM OF g)

① MINUS SIGN IS BY OBSERVATION: KINETIC SYSTEMS ARE ENERGETICALLY BOUNDED BELOW →

THIS IS DIFFERENT IN QUANTUM MECHANICAL SYSTEMS

② $\beta\sim\frac{1}{T}$, p SHOULD BE SMALL IF T IS SMALL

③ k FIXES UNITS ~ CONCEPT OF THERMAL ENERGY

(THINK OF BAROMETRIC FORMULA)

- THE PLANCK-SYSTEM IS ONE OF THE SIMPLEST QUANTUM SYST.

QUANTUM MECHANICS → SUPERPOSITION + INTERFERENCES

THERMAL PHYSICS → THERMAL ENERGY

THERE IS EQUIPARTITION IN SYSTEMS IN THERMAL EQUILIBRIUM

$\frac{\vec{p}^{\,2}}{2m}=\frac{3}{2}kT$ (FOR A CLASSICAL SYSTEM)

$c\vec{p}=\frac{3}{2}kT$ (FOR A RELATIVISTIC SYSTEM)

+ $p=\frac{h}{\lambda}=\hbar k$ de BROGLIE WAVE LENGTH

→ THERMAL WAVELENGTH $\lambda_{TH}=\frac{h}{p}=\frac{2}{3}\frac{hc}{k_B T}$

PARTICLE SEPARATION $\gg\lambda_{TH}$ INTERFERENCE NOT IMPORTANT

SYSTEM BEHAVES CLASSICALLY (TRUE AT HIGH ENERGIES)

PARTICLE SEPARATION $\ll \lambda_{th}$ INTERFERENCE IS IMPORTANT
SYSTEM BEHAVES QUANTUMMECHANICALY (TRUE AT LOW ENERGIES)

- PHOTON NUMBER STATISTICS

STATES i ~ SORTED IN ENERGY

WEIGHTS g_i ~ NUMBER OF PHOTONS EACH ~~STAT~~ STATE CAN CARRY

OCCUPATION n_i ~ NUMBER OF PHOTONS IN A STATE

$w(n_i;g_i)$ ~ NUMBER OF REALISATIONS FOR n_i PHOTONS IN STATE i WITH STATISTICAL WEIGHT g_i

$\prod_i w(n_i;g_i)$ ~ NUMBER OF WAYS FOR PUTTING $n=\sum n_i$ PHOTONS IN THE STATES i, n_i EACH.

CONSTRUCT $w(n_i,g_i)$: THERE CAN BE ARBITRALY MANY PHOTONS IN A SINGLE STATE.

$w(n,1)=1$ 1 WAY OF PUTING n-PHOTONS IN 1 STATE

$w(n,2)=n+1=\frac{(n+1)!}{n!}$ $n+1$ WAYS OF PUTTING n PHOTONS INTO 2 STATES

$$w(n,3)=\sum_k w(n-k,2)=\sum\frac{(n-k+1)!}{(n-k)!}=\frac{(n+2)!}{n!\cdot 2!}$$

$\vdots$

$$w(n,g)=\frac{(n+g-1)!}{n!(g-1)!}\approx\frac{(n+g)!}{n!\,g!}$$

$$\downarrow$$

$$\prod_i w(n_i;g_i)=\prod_i\frac{(n_i+g_i)!}{n_i!\,g_i!}=\Omega$$

$$\ln\Omega=\sum_i(n_i+g_i)\cdot\ln(n_i+g_i)-n_i\ln n_i-g_i\ln g_i$$

WITH THE STIRLING-APPROXIMATION $\ln(n!)=n\ln n$

- FIND THE MAXIMUM ~ MOST LIKELY CONFIGURATION UNDER THE CONDITIONS $\sum_i n_i = N$ (TOTAL NUMBER) AND $\sum_i n_i \epsilon_i = E$ (TOTAL ENERGY)

$$X = \ln \Omega + \alpha (N - \sum n_i) + \beta (E - \sum n_i \epsilon_i)$$

WITH TWO LAGRANGE-MULTIPLIERS α AND β

$$\rightarrow \frac{dX}{dn_i} = 0 = \ln(n_i + g_i) - \ln n_i - \alpha - \beta \epsilon_i$$

$$\rightarrow \frac{n_i + g_i}{n_i} = \exp(-\alpha - \beta \epsilon_i)$$

$$n_i = g_i \frac{1}{\exp(\alpha + \beta \epsilon_i) - 1} \quad \text{BOSE FACTOR}$$

COMPARISON TO E.G. IDEAL GAS SHOWS

$\alpha = \frac{\mu}{kT}$ ~ FUGACITY AND $\beta = \frac{1}{kT}$ BOLTZMANN FACTOR

- PHOTONS ARE ULTRARELATIVISTIC PARTICLES $\epsilon = cp$ 2 SPIN STATES

DISPERSION: $p = \frac{h}{\lambda} = \frac{\hbar \omega}{c}$

FUGACITY $\frac{\mu}{kT}$ (OR CHEMICAL POTENTIAL) $= 0$ FOR PHOTONS

$$n = 2 \cdot \int \frac{d^3p}{h^3} \frac{1}{\exp(\beta c p - 1)} = \frac{1}{\pi^2 \cdot c^3} \int_0^\infty d\omega \frac{\omega^2}{\exp(\beta \hbar \omega) - 1} = 4 \zeta(3) \left(\frac{kT}{\hbar c}\right)^3$$

$$u = 2 \cdot \int \frac{d^3p}{h^3} \cdot \frac{cp}{\exp(\beta c p - 1)} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{\exp(\beta \hbar \omega) - 1} = \frac{\pi}{15 (\hbar c)^3} (kT)^4$$

PLANCK SPECTRUM $S(\omega)$ $\quad u = \int_0^\infty d\omega \, S(\omega) \rightarrow$

$$\rightarrow S(\omega) = \frac{\hbar}{\pi^2 \cdot c^3} \cdot \frac{\omega^3}{\exp(\beta \hbar \omega) - 1}$$

TYPICAL INTEGRALS IN THIS CONTEXT

$$\int_0^\infty d\omega \frac{\omega^{n-1}}{\exp(\omega)-1} = \int_0^\infty d\omega \cdot \omega \cdot \frac{1}{\exp(\omega)-1} = \int_0^\infty d\omega\, \omega^{n-1} \sum_m \exp(-m\omega) =$$

$$= \sum_m m^{-n} \cdot \int_0^\infty dy\, y^{n-1} \cdot \exp(-y) = \zeta(n) \cdot \Gamma(n-1) \quad , \; y = m\omega$$

RAYLEIGH-JEANS-LIMIT $\hbar\omega \ll kT$

$$S(\omega) = \frac{\omega^3}{\exp(\beta\hbar\omega)-1} \longrightarrow \frac{\omega^3}{1+\beta\hbar\omega-1} \sim \omega^2$$

WIEN-LIMIT $\hbar\omega \gg kT$

$$S(\omega) = \frac{\omega^3}{\exp(\beta\hbar\omega)-1} \longrightarrow \omega^3 \cdot \exp(-\beta\hbar\omega)$$

DISPLACEMENT LAW

$$\frac{dS}{d\omega} = 0 \longrightarrow \frac{3-x}{e^x-1} = 0 \quad \text{WITH } x = \frac{\hbar\omega}{kT} = \beta\hbar\omega$$

$\sim$ SOLVED BY $x \simeq 2.81$, $\frac{\omega}{T} = \text{CONST.}$

WIEN-RADIATION LAW. THE (-1) IN THE BOSE FACTOR COMES FROM THE INDISTINGUISHABILITY OF PHOTONS IN THEIR COUNT. STATISTICS.

CLASSICAL PARTICLES JUST WOULD HAVE THE BOLTZMAN-FACTOR, E.G.

$$n = 2 \cdot \int \frac{d^3p}{h^3} \exp(-\beta cp) = \frac{1}{\pi^2 \cdot c^3} \cdot \int_0^\infty d\omega \cdot \omega^2 \cdot \exp(-\beta\hbar\omega) \sim T^3$$

THE BASIC INTEGRAL PROPERTIES ARE IDENTICAL, "ONLY" THE PREFACTORS DON'T MATCH.

C15 THOMSON-SCATTERING AND THE THOMSON CROSS SECTION

- PLACE A TEST CHARGE IN THE WAY OF AN INCOMING ELECTROMAGNETIC WAVE ~ RADIATION PRESSURE EXPERIENCED BY THE CHARGE + SCATTERING OF THE INCIDENT RADIATION.

MOTION OF THE ELECTRON: IN THE NON-RELATIVISTIC LIMIT

$$m \cdot \frac{d^2}{dt^2}\vec{x} = q \cdot (\vec{E} + \vec{v} \times \vec{B}) \;;\; q = -e \text{ FOR ELECTRON}$$

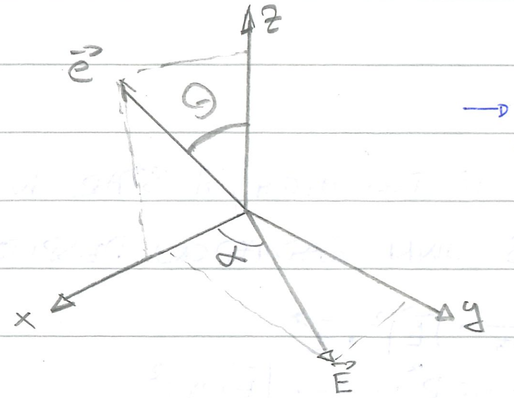
FOR VERY SMALL VELOCITIES $|\vec{E}| \gg |\vec{v} \times \vec{B}| \rightarrow$

$$c \cdot \dot{\vec{\beta}} = \frac{q}{m}\vec{E} \quad \text{WITH} \quad \vec{\beta} = \frac{\vec{v}}{c}$$

RADIATION EMITTED BY THE TEST CHARGE

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c} \cdot |\vec{e} \times \dot{\vec{\beta}}| \quad \text{INTO THE DIRECTION } \vec{e}$$

$$\rightarrow \frac{dP}{d\Omega} = \frac{q^4}{4\pi m^2 c^3} |\vec{e} \times \vec{E}|^2 \quad \text{WITH THE LORENTZ-FORCE SUBSTITUTED}$$



$$\rightarrow \frac{dP}{d\Omega} = \frac{q^4 E^2}{4\pi m^2 \cdot c^3} \cdot (1 - \cos^2\alpha \sin^2\Theta)$$

- THOMSON CROSS SECTION: RATIO BETWEEN SCATTERED AND INCOMING RADIATION

$$\vec{S} = \frac{c}{4\pi} |\vec{E}|^2 \cdot \vec{e}_z$$

$$\frac{d\sigma}{d\Omega} = \frac{dP}{S \cdot d\Omega} = \frac{q^4}{(mc^2)^2} \cdot (1 - \cos^2\alpha \sin^2\Theta)$$

- DIFFERENTIAL CROSS-SECTION
  - ELECTROSTATIC ENERGY OF A HOMOGENEOUSLY CHARGED SPHERE

$$=$$

REST MASS ENERGY (RELATIVISTIC)

$\frac{e^2}{r_e} = m c^2$ DETERMINES THE CLASSICAL ELECTRON RADIUS

$$r_e = \frac{e^2}{mc^2}$$

$\rightarrow \frac{d\sigma}{d\Omega} = r_e^2 \cdot (1 - \cos^2\alpha \sin\Theta)$

↳ "AREA" ASSOCIATED WITH THE ELECTRON, $\sim 10^{-28}\, m^2$

- AVERAGE OVER ALL POLARISATION STATES AND DIRECTIONS

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2\pi} \cdot \int_0^{2\pi} d\alpha \cdot \frac{d\sigma}{d\Omega}(\alpha) = \frac{r_e^2}{2}(1+\cos^2\Theta)$$

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \pi \cdot r_e^2 \cdot \int_{-1}^{+1} d\mu (1+\mu^2) = \frac{8\pi}{3} \cdot r_e^2 = \sigma_T$$

$\sigma_T \simeq 10^{-28}\, m^2$ ~ TOTAL THOMSON CROSS-SECTION

EVENT RATE = PARTICLE FLUX · CROSS-SECTION

- EDDINGTON-LIMIT

IF THE RADIATION PRESSURE IS TOO HIGH, A STAR WOULD BE BLOWN APART BY ITS OWN RADIATION DESPITE GRAVITY

MOMENT DENSITY $\frac{\vec{S}}{c} = p = \frac{1}{4\pi} \cdot |\vec{E}|^2 \cdot \vec{e}$

TOTAL LUMINOSITY $L = \int d\vec{A} \cdot \vec{S} = 4\pi R^2 \cdot \frac{c}{4\pi} \cdot |\vec{E}(R)|^2$

$\longrightarrow \frac{\vec{S}}{c} = \frac{L}{4\pi R^2 \cdot c} \vec{e}$

FORCE ON AN ELECTRON $\vec{F}_S = \frac{\vec{S}}{c} \cdot \sigma_T = \frac{L \cdot \sigma_T}{4\pi \cdot R^2 \cdot c} \cdot \vec{e}$

VS. GRAVITY $\vec{F}_g = -\frac{GmM}{R^2} \cdot \vec{e}$

FORCE BALANCE DEFINES EDDINGTON LIMIT

$$L_{EDD} = \frac{4\pi G M m_i c}{\sigma_T} \simeq 33 \cdot 10^3 \cdot L_\odot \left(\frac{M}{M_\odot}\right)$$

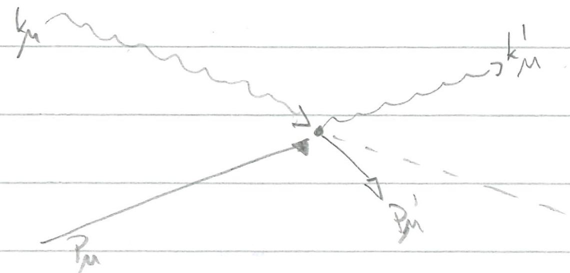
m_i SHOULD BE THE MASS OF THE IONS WHICH ARE TIGHTLY ~~COUPLE~~ COUPLED TO THE ELECTRONS.

- COMPTON SCATTERING

- MOMENTUM TRANSFORM FROM A PHOTON ONTO AN ELECTRON

PHOTON $\quad ck^\mu = \omega \cdot \begin{pmatrix} 1 \\ \vec{e} \end{pmatrix}$

ELECTRON $\quad p^\mu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = \gamma m \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$



CONSERVATION OF 4-MOMENTUM

$$p^\mu + \hbar k^\mu = p'^\mu + \hbar k'^\mu$$

$$\begin{cases} E + \hbar\omega = E' + \hbar\omega' & \mu = 0 \\ c\vec{p} + \hbar\omega\vec{e} = c\vec{p}\,' + \hbar\omega'\vec{e}\,' & \mu = 1;2;3 \end{cases} \quad \text{| SPHERE}$$

$$E'^2 = E^2 + 2\hbar c\vec{p} \cdot (\omega\vec{e} - \omega'\vec{e}\,') + \hbar^2 \cdot (\omega\vec{e} - \omega'\vec{e}\,')^2$$

$(E^2 = c^2p^2 + m^2c^4$, AND SUBSTITUTE $\vec{p}\,')$

USE ENERGY CONSERVATION TO ELIMINATE E'

$$E(\omega - \omega') = \hbar\omega\omega' \cdot (1 - \cos\Theta) + c\vec{p}(\omega\vec{e} - \omega'\vec{e}\,'), \quad \Theta = \sphericalangle(\vec{e}, \vec{e}\,')$$

ELECTRON AT REST: $\vec{p} = 0$; $E = mc^2 \rightarrow$

$$\frac{\omega'}{\omega} = \frac{1}{1 + \epsilon(1 - \cos\Theta)} \quad \text{WITH } \epsilon = \frac{\hbar\omega}{mc^2}$$

• COMPUTE AVERAGE CHARGE IN ENERGY (OVER ALL ANGLES)

$$\frac{\langle \Delta E \rangle}{E} = \frac{\langle \omega' \rangle - 1}{\omega} = \frac{1}{\sigma_T} \cdot \frac{r_e^2}{2} \cdot \int \sin\theta \, d\theta \, d\varphi \left(\frac{1+\cos\theta}{1+\epsilon(1-\cos\theta)} - 1 \right) =$$

$$= \frac{\pi \cdot r_e^2}{\sigma_T} \cdot \left(\frac{1}{\epsilon^3} \cdot \left(\ln(1+2\epsilon)(2\epsilon^2+2\epsilon+1) - 2\epsilon(1+\epsilon) \right) - 1 \right)$$

IF $\epsilon \ll 1$ (LOW ENERGY LIMIT)

$$\frac{\langle \Delta E \rangle}{E} \simeq \frac{8\pi}{3} \frac{r_e^2}{\sigma_T} \cdot (1-\epsilon-1) \simeq -\frac{\hbar\omega}{mc^2} \simeq -\epsilon$$