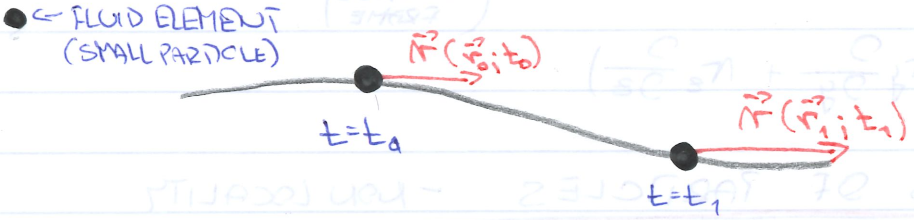
THEORETICAL ASTROPHYSICS

A1 FLUID MECHANICS

- FIRST WE WILL TALK ABOUT FLUID MOTION, CONTINUUM (FIELD) THEORY
- CONTINUUM WILL BE DESCRIBED BY DENSITY $\rho(\vec{r};t)$ AND VELOCITY $(\vec{r};t)$. MASS WILL BE SMOOTHLY DISTRIBUTED.

- FLUID ELEMENT (CENTRAL CONCEPT) - LET'S IMAGINE WE HAVE A CLOUD OF PARTICLES (CLOUD SUFICENTLY LARGE)
  - ELEMENT IS LARGE ENOUGH TO CONTAIN MANY PARTICLES
  - WE WILL DESRIBE VARIATION IN DENSITY AND VELOCITY, SO IT HAS TO BE SMALL ENOUGH COMPARED TO LENGTH
  - SMALL VELOCITIES $|\vec{v}| \ll c$ (NO RELATIVITY)
  - GALILEAN RELATIVITY
    - ADDITIVITY OF VELOCITIES
    - UNIVERSAL TIME
  - SMALL POTENTIALS - GRAVITATIONAL POTENTIAL $|\phi| \ll c^2$

A2 EULER & LAGRANGE FRAME



TWO WAYS HOW TO DESCRIBE THIS:

1) EULER FRAME - FIXED LABORATORY COORDINATES, WE ARE STANDING ON A SHORE (BRIDGE). WE HAVE VELOCITIES $\underline{v(\vec{r};t)}$ AT ALL POSSITION $\underline{\vec{r}}$ AT A FIXED TIME
(SNAPSHOT)

2) LAGRANGE FRAME - WE OBSERVE WHILE/MOVING WITH THE FLOW, $\underline{\vec{v}(\vec{r}_{(t)};t)}$

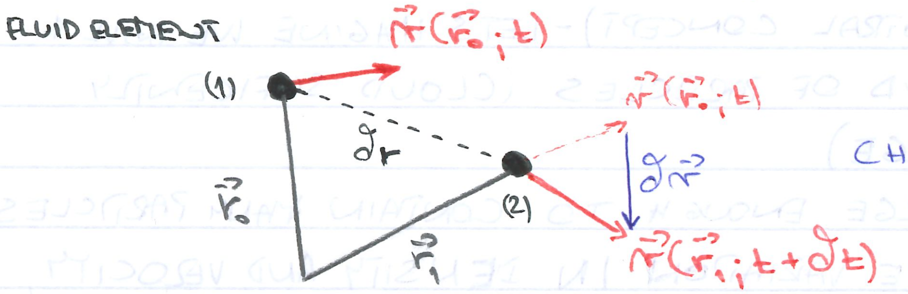
A3 ACCELERATION OF A FLUID

- FIRST LAGRANGE FRAME - WE FEEL ACCELERATION WE MOVE TO A PLACE WHERE THE VELOCITY IS LARGER (EXPLICIT VARIATIONS OF $\vec{v}$)

$$\delta\vec{v}=\vec{v}(\vec{r}_0;t+\delta t)-\vec{v}(\vec{r}_0;t)$$

- SECOND EULER FRAME - ONE "STATIC" PLACE, AND FLUID MOVES ALONG, BUT LATER THE HIGER VELOCITY APPEARS. FLUID ELEM. MOVES TO A NEW POSSITION

$$\delta\vec{v}=\vec{v}(\vec{r}_1;t+\delta t)-\vec{v}_0(\vec{r}_0;t+\delta t)$$



(CHANGE IN A VELOCITY

HOW THE ACCELERATION LOOKS LIKE:

$$\delta\vec{v}=\vec{v}(\vec{r}_1;t+\delta t)-\vec{v}(\vec{r}_0;t)\overset{\text{TAYLOR EXPAN}}{=}\frac{\partial\vec{v}}{\partial t}\delta t+\frac{\partial\vec{v}}{\partial\vec{r}}\delta\vec{r}$$

$$\frac{d\vec{v}}{dt}=\lim_{\delta t\to 0}\frac{\delta\vec{v}}{\delta t}=\frac{\partial\vec{v}}{\partial t}+\frac{\partial\vec{v}}{\partial\vec{r}}\cdot\underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}}=\overbrace{\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)}^{\text{ADVECTIVE DERIVATIVE}}\vec{v}$$

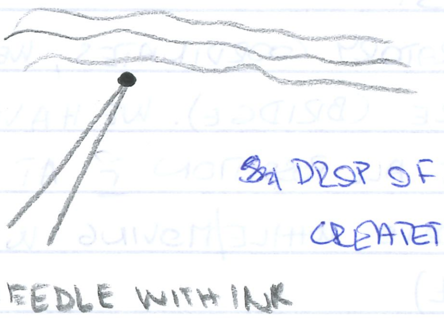
ACCELEROMETER (LAGRANGE FRAME)

FLOW BRINGS "ACCELERATION" (EULER FRAME)

(NON-LINEARITY)

$$\vec{v}\cdot\nabla=\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)$$

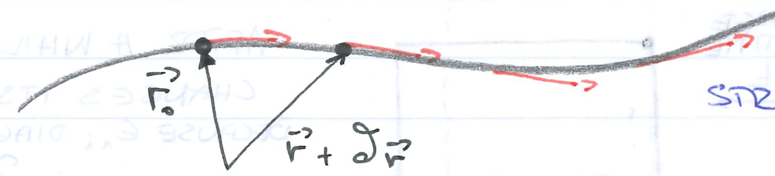
A4 TRAJECTORIES OF PARTICLES - NON LOCALITY



~~Sa~~ A DROP OF INK FOLLOWS THE STREAM AND CREATETS STREAK LINE (STREAM LINE)

$$\underbrace{\vec{r}(t) - \vec{r}(t_0)}_{\vec{r}_0} = \int_{t_0}^{t} dt' \cdot \underbrace{\vec{v}(\vec{r}_0; t)}$$

LAGRANGE VELOCITY - THIS WHAT DROP OF INK FEELS WHEN IT'S ADDED TO THE WATER



STREAM LINES - FLOW PATTERN AT FIXED t

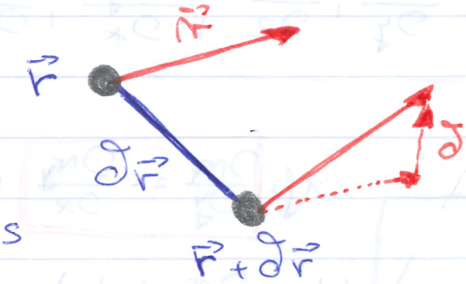
A5 DEFORMATION OF THE FLUID ELEMENTS DUE TO VELOCITY VARIATIONS

AT FIXED t: ① A FLUID ELEMENT AT $\vec{r}$ WITH VELOCITY $\vec{v}(\vec{r};t)$

② A FLUID ELEMENT AT $\vec{r}+\partial\vec{r}$ WITH VELOCITY $\vec{v}+\partial\vec{v}$

$$\partial\vec{v} = (\partial\vec{r}\cdot\nabla)\vec{v}$$

IN COMPONENTS $dv_i = \overbrace{\frac{\partial v_i}{\partial x_j}}^{\text{VELOCITY TENSOR}} dx_j$



$\partial\vec{v}$ - DIFFERENTIAL MOTION OF THE DIST. ELEMENT

TWO FLUID ELEMENTS

VELOCITY TENSOR DESCRIBES IN WHAT WAY FLUID ELEMENT SQUIRE..., RELATIVE MOTION IS DESCRIBED BY THE VELOCITY TENSOR $\frac{\partial v_i}{\partial x_j}$ IT CAN BE DECOMPOSED:

1) SYMMETRIC PART $\epsilon_{ij} = 1/2\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, $\epsilon_{ij} = \epsilon_{ji}$
(SHEAR) (SMYKOVÉ?)

2) ANTISYMMETRIC PART $\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$, $\omega_{ij} = -\omega_{ji}$
(VORTICITY) (VÍROVOST?)

IN SYMMETRIC PART WE CAN TAKE A LOOK AT THE TRACE:

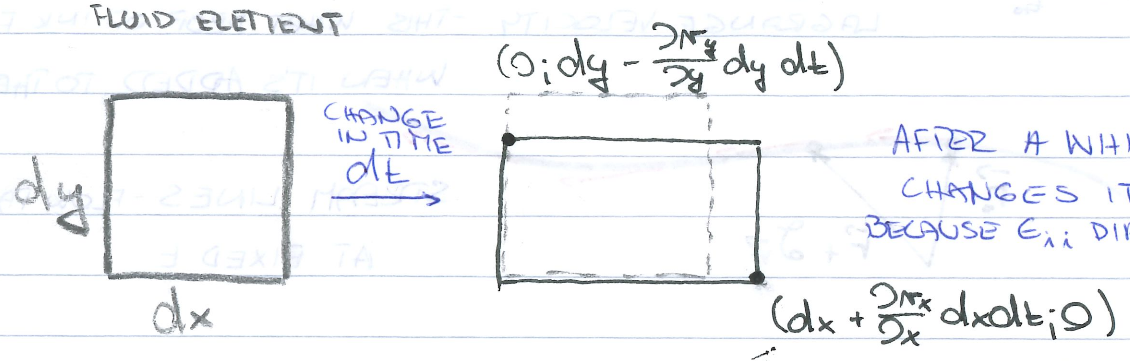
$\epsilon_{ii} = \frac{\partial v_i}{\partial x_i} = \text{div}\,\vec{v} = \nabla\cdot\vec{v}$ → RATE OF CHANGE OF A FLUID ELEMENT.

(NESTLAČITELNÁ) ← NIC S TLAKEM

IF $\text{div}\,\vec{v} = 0$ THEN THE FLOW IS IN~~OPO~~COMPRESIBLE
↳ VOLUME OF THE FLUID DOES NOT CHANGE

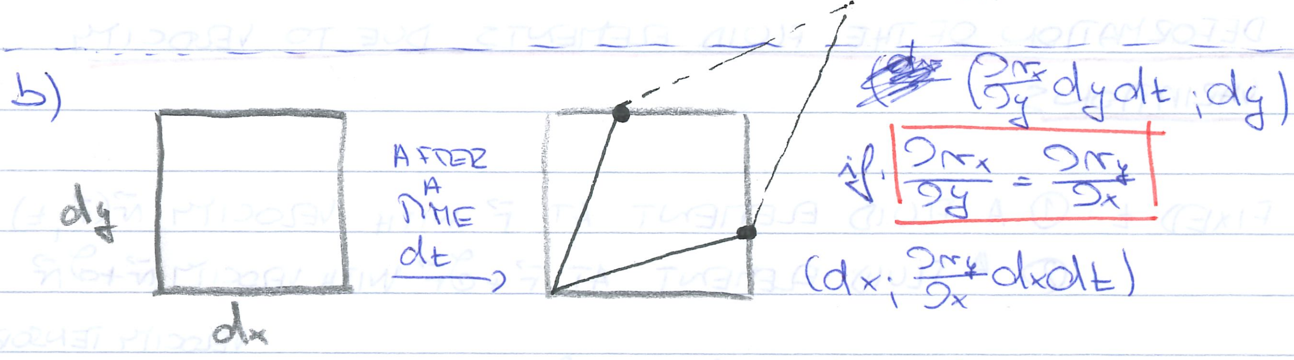
A6 VISUALI~~Z~~SATION OF THE VELOCITY TENSOR

a)



AFTER A WHILE (dt) ELEMENT CHANGES ITS SHAPE (VOLUME) BECAUSE ϵ_{ii} DIAG. ELEMENTS: $\frac{\partial v_x}{\partial x}; \frac{\partial v_y}{\partial y}; \frac{\partial v_z}{\partial z}$ ③

b)

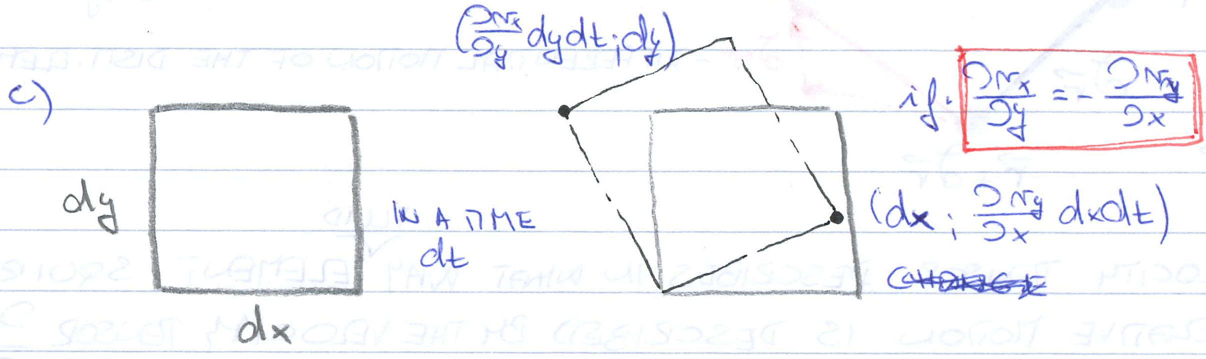


if: $\boxed{\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}}$

⑥

ϵ_{ij}: OFF DIAGONAL ELEMENTS: $\frac{\partial v_x}{\partial y}, \frac{\partial v_x}{\partial z}, \frac{\partial v_y}{\partial x}, \frac{\partial v_y}{\partial z}, \frac{\partial v_z}{\partial x}, \frac{\partial v_z}{\partial y}$
(SHEAR)

c)



if: $\boxed{\frac{\partial v_x}{\partial y} = - \frac{\partial v_y}{\partial x}}$

ω_{ij}: OFF DIAGONAL ELEMENTS
(VORTICITY)

b) & c) BOTH CHANGE SHAPE BUT CONSERVE VOLUME, BOTH ϵ_{ij}, ω_{ij} AND div($\vec{v}$) HAVE UNITS OF ANGULAR VELOCITIES.

A7 DECOMPOSITION OF THE VELOCITY TENSOR

MOTION $\vec{r}(t+\Delta t) = \vec{r}_0(t) + \vec{v}\Delta t$

RELATIVE MOTION $\frac{\partial r_i}{\partial x_j} \approx \underbrace{\delta_{ij} + \frac{\partial v_i}{\partial x_j}\Delta t}_{\text{TOHLE JE ZACÁTEK EXPONENCIELY}} + O(\delta t^2) + \ldots = \exp(\frac{\partial v_1}{\partial x_j}\Delta t)$

HOW MY NEIBORING POINT CHANGE'S ITS POSSITION IN DIFFERENT RATE THAN I DO.

CHANGE IN VOLUME $d^3x(t+\Delta t) = \underbrace{\det\left(\frac{\partial r_i}{\partial x_j}\right)}_{\text{FUNCTIONAL DETERMINANT}} \cdot d^3x(t)$

HANDY NOTE,

$$\ln \det A = \ln \prod_i \overset{\text{VLASTNÍ HODNOTY}}{\lambda_i} = \sum_i \ln\lambda_i = \underset{\text{STOPA - TRACE}}{\mathrm{tr}} \ln A$$

$$\ln \det\left(\frac{\partial r_i}{\partial x_j}\right) = \mathrm{tr}\ln\left(\frac{\partial r_i}{\partial x_j}\right) = \mathrm{tr}\left(\frac{\partial v_i}{\partial x_j}\right)\cdot \Delta t = \frac{\partial v_i}{\partial x_i}\Delta t = \mathrm{div}(\vec{v})\,\Delta t$$

→ INCOMPRESSIBLE FLOWS CONSERVE VOLUMES

VELOCITY TENSOR HAS A BASIS IN TERMS OF THE GELL-MANN MATRICES WHICH ARE ALL TRACELESS; VOLUMES ARE CONSERVED BY SHEAR AND VORTICITY

ONLY $\mathrm{div}\,\vec{v}$ CHANGES VOLUMES

$\frac{\partial v_i}{\partial x_j}$ GM BASIS SPLITS IN TWO

- → $\epsilon_{ij} = +\epsilon_{ji}$ 5 TRACELESS SYMETRIC MATRICES (SHEAR)
- → $\omega_{ij} = -\omega_{ji}$ 3 TRACELESS ANTISYMETRIC MATRICES (VORTICITY
- → 1 UNIT MATRIX, CHANGE IN VOLUMES, $\mathrm{tr}(\;) = \underline{\mathrm{div}\,\vec{v}}$

$$\lambda_z = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix} \qquad \lambda_y = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ -1 & & 0 \end{pmatrix} \qquad \lambda_x = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}$$

$$\mathrm{rot}(\vec{v})_z = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \qquad \mathrm{rot}(\vec{v})_y = \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\,; \quad \mathrm{rot}(\vec{v})_x = \frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}$$

$\omega_{ij} = -\omega_{ji}$ HAS 3 INDEPENDENT AND IS ANTISYMETRIC

- CONTRACT WITH ϵ_{ijk} AND OBTAIN AN AXIAL VECTOR

$\omega_i = \epsilon_{ijk}\,\omega_{jk}$, $\vec{\omega} = \mathrm{rot}(\vec{v}) = \nabla \times \vec{v}$

A8 EQUATION OF MOTION -DYNAMICS

ACCELERATION OF FLUID ELEMENTS

- PREASSURE GRADIENTS ∇p
- GRAVITY $\nabla \Phi$
- ELECTRIC & MAGNETIC FORCES
- VISCOUS FORCES

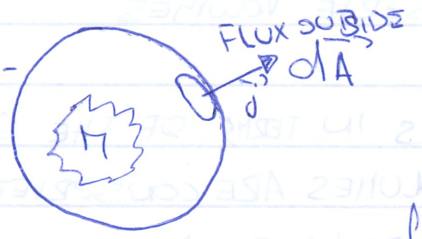
EULER

NAVIER STOKES

A9 CONTINUITY EQUATION AND THE CONSERVATION OF MASS, VOLUME V, SURFACE ∂V, CONTAINS MASS m INSIDE

DECAY, CHEMICAL REACTION

THAT'S HOW IT GET'S OUT



CHANGE IN MASS

$$\frac{dM}{dt} = -\frac{d}{dt}\int d^3x\, \rho = -\int d\vec{A}\cdot\vec{j} =$$

$$= \int_V d^3x\, \mathrm{div}\,\vec{j} \longrightarrow \frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0$$

j ~ MATTER FLUX (OUT WARD) $j = \rho\cdot\vec{v}$

ρ ~ DENSITY

APPLYING GAUSS THEOREM

$$\int_V dV\left(\frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j}\right) = 0$$

IF THE FLUX THROUGH THE SURFACE IS THE ONLY WAY IN WHICH M CAN CHANGE

$$\vec{j} = \rho\cdot\vec{v} \Rightarrow \mathrm{div}(\vec{j}) = \mathrm{div}\,\rho\cdot\vec{v} + \rho\cdot\mathrm{div}(\vec{v})$$ IN GENERAL

AGAIN EQ. OF CONTINUITY:

$$\frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\vec{v}) = 0$$

INCOMPRESIBLE

$$\frac{\partial}{\partial t}\rho + \vec{v}\cdot\nabla\rho = 0$$

COMPRESSIBLE

$$\underbrace{\frac{\partial}{\partial t}\rho + \vec{v}\cdot\nabla\rho}_{\text{ADVECTIVE DERIVATIVE}} = -\rho\,\mathrm{div}(\vec{v})$$

CONTINUITY IS NONLINEAR

(BUT IN ED IS LINEAR)

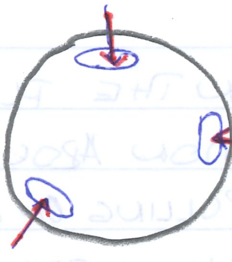
A10 KINEMATICS

$$\vec{a} = \frac{d\vec{v}}{dt} = \underbrace{\frac{\partial}{\partial t}\vec{v} + (\vec{v}\nabla)\vec{v}}_{\text{2 CONTRIBUTION}} = \sum$$ SPECIFIC FORCES ACTING OF FLUID ELEMENT (FORCE DENSITIES)

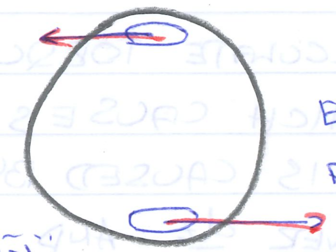
LAGR. & EULER FRAME

A11 PRESSURE FORCES AND VISCOUS FORCES
(STRESS & PREASURE)

- THESE FORCES ACT ON A SURFACE OF THE ELEMENT



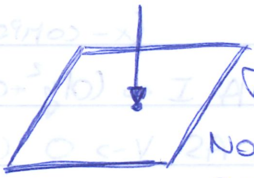
ISOTROPIC PREASURE (FEELS PREASURE FROM ALL SIDES)



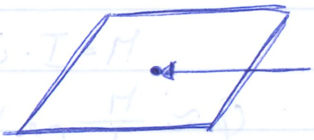
MANY FLUID ELEMENTS EXPERIENCE SHEARING FORCES WITH SOUROUNDING FLUID

FLUID ELEMENT TRAPPED IN FLOW (VISCOUS FORCES)

STRESS TENSOR → SURFACE ELEMENT $d\vec{A}$
→ 3 COMPONENTS OF FORCE FOR 3 COMPONENTS OF $d\vec{A}$.



σ_{ii} NORMAL FORCES



σ_{ij} $i \neq j$

TANGENTIAL FORCES

σ_{ij} - FORCE DENSITY IN i-DIRECTION ACTING ON A SURFACE ELEMENT IN j-DIRECTION.

$d\vec{F} = \sigma \cdot d\vec{A} \rightarrow \frac{d\vec{F}}{dA} = \sigma \cdot \vec{n}$ WITH $d\vec{A} = \vec{n}\, dA$

σ - LINEAR TRANSFORMATION BETWEEN ORIENTATION $\vec{A}$ AND FORCE $\vec{F}$.

σ_{ij} IS SYMETRIC $\sigma_{ij} = \sigma_{ji}$ (ORIGIN OF THIS IS IN RELATIVITY) THIS SYMERY LINKS TO A VELOCITY TENSOR ϵ_{ij}

ISOTROPIC PART OF σ_{ij} ~ PRESSURE

$$\sigma_{ij} = \underbrace{\overset{\text{PRIME}}{\sigma_{ij}'}}_{\text{VISCOSITY}} - \underbrace{p\delta_{ij}}_{\text{PRESSUR}}$$

p - IS PUSHING AGAINST $d\vec{A}$ -> MINUS SIGN

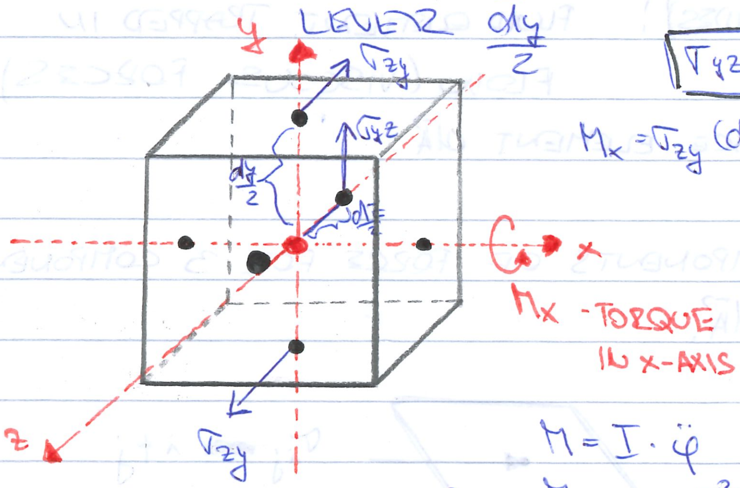
AND IS ISOTROPIC

σ_{ij}' CAN STILL HAVE ELEMENTS ON THE DIAGONAL

$$tr(\sigma) = \sigma_{ii} = -3p \qquad tr(\sigma') = \sigma_{ii}' = 0$$

$\sigma_{ij} = \sigma_{ji}$ -> CALCULATE TORQUE M_x ON THE FLUID ELEMENT WHICH CAUSES A ROTATION AROUND THE X-AXIS

M_x IS CAUSED BY σ_{yz} PULLING ON $dxdy$ WITH LEVER $\frac{dz}{2}$ AND BY σ_{zy} PULLING ON $dxdz$ WITH LEVER $\frac{dy}{2}$



$\boxed{\sigma_{yz} \,\&\, \sigma_{zy} \text{ GENERATE TORQUE}}$

$$M_x = \sigma_{zy}(dxdz)\frac{dy}{2} - \sigma_{yz}(dxdy)\frac{dz}{2} = (\sigma_{zy} - \sigma_{yz})\frac{dV}{2}$$

SYSTEM STARTS ROTATING.

(x - COMPONENT)

$M = I \cdot \ddot{\varphi}$ WITH INERTIA $I \sim (dy^2 + dz^2)dV$

$\ddot{\varphi} \approx \frac{M}{I} \sim V^{-2/3}$ ——> AND ∞ AS $V \to 0$ (GOES TO ZERO)

THEREFORE $\sigma_{ij} = \sigma_{ji}$, STRESS TENSOR IS SYMETRIC OTHERWISE SMALL FLUID ELEMENTS WOULD START SPINNING.

A12 VISCOSITY

FOCUS ON NEWTONIAN FLUIDS: σ_{ij} DEPENDS LINEARLY AND INSTANTENEOUSLY ON THE (LIKEWISE SYMETRIC) VELOCITY TENSOR ϵ_{ij}

NEWTONIAN FLUID - DRY PAINT, KETCHUP

$$\sigma'_{ij} = 2\eta\left(\epsilon_{ij} - \frac{1}{3}tr(\epsilon)\cdot\delta_{ij}\right) + \xi\, tr(\epsilon)\,\delta_{ij}$$ LAMÉ ANZAC

η - SHEAR VISCOSITY
ξ - BULK VISCOSITY

η: $\sigma'_{ij} = \eta\cdot\frac{\partial v_i}{\partial x_j}$ $\quad\xi$: $\sigma'_{ii} = \xi\cdot\frac{\partial v_i}{\partial x_i} = \xi\, div(\vec{v})$

INCOMPRESSIBLE FLUIDS $\sigma'_{ij} = 2\eta\epsilon_{ij}$, BECAUSE $div(\vec{v}) = 0$

η & ξ - LAMÉ VISCOSITY COEFFICIENTS, SPECIFIC TO ANY SUBSTANCE

IN INCOMPRESSIBLE FLUIDS, KINEMATICS IS NOT CHANGING A VOLUME OF FLUID ELEMENT (WATER).

A13 NAVIER - STOKES EQUATION

- APPLYING NEWTONIAN EQ. OF MOTION TO A FLUID ELEMENT WHICH TRANSPORTS THE MOMENTUM $\int dV\rho\vec{v}$

$$\frac{d}{dt}\underbrace{\int_V V\rho\vec{v}}_{\text{TOTAL MOMENTUM}} = \int_V dV\rho\vec{f} + \int_{\partial V} d\vec{A}\cdot\sigma$$

$\rho\vec{v}$ - MOMENTUM DENSITY

2 TYPES OF FORCES <
$\vec{f}$ - BODY FORCES ACTS ON A VOLUME
σ - SURFACE FORCES ACTS ON A SURFACE OF A VOLUME

- TIME DERIVATIVE $\frac{d}{dt}$ ACTS ONLY ON ~~A~~ THE VELOCITY BECAUSE IN THE LAGRANGE - PICTURE THE FLUID ELEMENT ALWAYS CONTAINS THE SAME ATOMS/MOLECULES: $dV\rho$ IS CONSTANT. THERE IS NO LOSS/GAIN IN THE FLUID.

$$\frac{d}{dt}\int dV(\rho\vec{v}) \Rightarrow \int_V dV\rho\frac{d\vec{v}}{dt}$$

- REWRITE THE SURFACE TERM AS A VOLUME INTEGRAL: GAUSS-THEOR...

$$\left(\int_{\partial V} d\vec{A}\cdot\sigma\right)_i = \int_{\partial V} dA\,\sigma_{ij} n_j = \int_V dV\frac{\partial\sigma_{ij}}{\partial x_j} = \left(\int_V dV\,\nabla\cdot\sigma\right)_i$$

$$\int_V dV \rho \frac{d\vec{v}}{dt} = \int_V dV \rho \vec{f} + \int_V dV \nabla \cdot \sigma \rightarrow \boxed{\rho \frac{d\vec{v}}{dt} = \rho \vec{f} + \nabla \sigma}$$

COSMETIC CHANGES

σ - CONTAINS PREASURE AND VISCOSITY

- **SEPARATING σ** , σ' - SIGMA PRIME ; PREASURE - p

$$(\nabla \cdot \sigma)_i = (\nabla \cdot \sigma')_i - \frac{\partial}{\partial x_j}(p \cdot \delta_{ij}) = (\nabla \sigma'_i - \nabla p)_i$$

- SUBSTITUTE FOR THE ADVECTIVE DERIVATIVE

$$\underbrace{\frac{\partial}{\partial t}\vec{v} + (\vec{v}\nabla)\vec{v}}_{\text{SOURCE OF NONLINEARITY}} = \underbrace{\vec{f}}_{\text{BODY (GRAVITATIONAL)}} - \underbrace{\frac{\nabla P}{\rho}}_{\text{PREASSURE}} + \underbrace{\frac{1}{\rho}\nabla\sigma}_{\text{VISCOSITY}}$$

IN ASTROPHYSICS : $\vec{f} = -\nabla\phi$; ϕ - GRAVITATIONAL POTENTIAL

↖ DEFINITION OF THE GRAV. POT. AND GRAV. FORCE PREASSURE ACT IN THE OPPOSITE DIRECTION OF THE SURFACE ELEMENT

MAXWELL EQ - LINEAR, SCALLED SYS. IS STILL THE SAME

A14 HYDROSTATIC EQUILIBRIUM

$$\left.\begin{array}{ll} \text{STATIONARY} & \vec{v} = 0 \\ \text{STATIC} & \partial_t \vec{v} = 0 \end{array}\right\} -\nabla\phi - \frac{\nabla P}{\rho} = 0$$

FROM NAVIER-STOKES EQUATION

(EARTH ATMOSPHERE)

IDEAL GAS $pV = NkT \Rightarrow P = \frac{\rho}{\{[(m)]\}}$ → FOR FIXED TEMPERATURE $P \simeq \rho$

$$\frac{\nabla P}{\rho} = \frac{\nabla \rho}{\rho} = \nabla \ln \rho = -\nabla\phi \rightsquigarrow \rho \propto e^{(-\phi)} = e^{(-gh)}$$

BAROMETRIC FORMULA

IN A HOMOGENEOUS GRAVITATIONAL FIELD : $\phi = gh$ →

→ BAROMETRIC FORMULA $p \propto e^{(-gh)}$

MARS - THIN ATMOSHERS - ALMOST NO PARACHUTES

VENUS - SIMILAR TO EARTH BUT HUGE PREASSURE, THIS IS CONNECTED TO THE HEAVY MOLECULS IN VENUS ATHMOSPHERE ⇒ HIGH PREASSURE AND DENSITY.

YOUNG EARTH - A LOT OF WATER IN THE ATMOSPHERE ⇒ HIGH PREASSURE, BUT THAN RAINED (FOR A LOT OF $10^{2(3)}$ YEARS).

A 15 VISCOSITY

LAMÉ ANZAC

$$(\nabla\cdot\sigma')_i = \frac{\partial\sigma_{ij}}{\partial x_j} = \eta\frac{\partial^2 v}{\partial x_i\partial x_j} + \left(\xi + \frac{\eta}{3}\right)\cdot\frac{\partial}{\partial x_i}\frac{\partial v_k}{\partial x_k}$$

SUBSTITUTE A DEFINITION OF ϵ_{ij}

$$\nabla\cdot\sigma = \underbrace{\eta\cdot\Delta\vec{v}}_{\text{SHEAR VISCOSITY}} + \underbrace{\left(\xi+\frac{\eta}{3}\right)\cdot\nabla \operatorname{div}\vec{v}}_{\text{COMPRESIBILITY} = 0 \text{ IF } \operatorname{div}(\vec{v})=0}$$

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\nabla)\vec{v} = -\vec{f} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\cdot\Delta\vec{v}$$

NAVIER - STOKES - EQUATION FOR INCOMPRESSIBLE FLUIDS

η ~ VISCOSITY ; $\frac{\eta}{\rho}$ ~ KINEMATIC VISCOSITY $\equiv \mu$

UP TO NOW THERE IS NO MICROSCOPIC THEORY OF VISCOSITY, IN EXAMPLE OF THE EFFECTIVE MOMENTUM TRANSPORT BETWEEN.

A16 FLUID MECHANICAL SIMILARITY

FLUID MECHANICAL IS CLASSICAL → NO SCALE ASSOCIATED WITH IT (NO $\hbar$; k; c...)

THERE ARE CERTAIN DIFFERENT TYPES OF SOLUTION + REGIMES BUT CAN ONE FIND SIMILAR SOLUTIONS FOR DIFFERENT TYPES OF FLUIDS AND GEOMETRICS?

MAXWELLOVY - LINEÁRNÍ - VED

NAVIER-STOKES - NELINEÁRNÍ NELZE PŘEŠKÁLOVAT, ALE
NAPŘ. VE VĚTNÉM TUNELU - COOL AIR
POTOM LEPŠÍ! ADJUSTING VISCOSITY OF AIR

DIMENSIONLESS FORM OF THE NAVIER-STOKES EQ

(SCALLING TRANSFORMATION)
((REMOVING UNITS))

$x \rightarrow x^* = \frac{x}{L}$ WITH LENGTH SCALE L

$t \rightarrow t^* = \frac{t}{T}$ WITH TIME SCALE T

$v \rightarrow v^* = \frac{v}{V}$ WITH VELOCITY V

$p \rightarrow p^* = \frac{p}{P}$ WITH PREASSURE P

$g \rightarrow g^* = \frac{g}{G} = \frac{\nabla \Phi}{G}$ WITH GRAVITY G

DERIVATIVES BEHAVES LIKE THIS

$$\frac{\partial}{\partial t} = \frac{\partial t^*}{\partial t} \frac{\partial}{\partial t^*} = \frac{1}{T} \frac{\partial}{\partial t^*}$$

$$\frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x} \frac{\partial}{\partial x^*} = \frac{1}{L} \frac{\partial}{\partial x^*}$$

RESCALED NAVIER-STOKES

BOTH

RESCALE ALL VARIABLES

RESCALE ALL THE DERIVATIVES

$$\frac{\rho V}{T} \frac{\partial \vec{v}^*}{\partial t^*} + \frac{\rho V^2}{L} (\vec{v}^* \nabla^*) \vec{v}^* = - \frac{P}{L} \nabla^* p^* - \frac{\rho G}{L} \nabla^* \phi^* + \frac{\eta \cdot V}{L^2} \Delta^* v^* \Big/ \frac{L}{\rho V^2}$$

$$\underbrace{\frac{L}{TV}}_{\text{STROUHAL NUMBER } (St^{1})} \cdot \frac{\partial \vec{v}^*}{\partial t^*} + (\vec{v}^* \nabla^*) \vec{v}^* = - \underbrace{\frac{P}{\rho V^2}}_{\text{EULER NUMBER } [Eu^{1}]} \cdot \nabla^* p^* - \underbrace{\frac{G}{V^2}}_{\text{FROUDE NUMBER } [Fr^{-2}]} \nabla^* \phi^* + \underbrace{\frac{\eta}{\rho V L}}_{\text{REYNOLDS NUMBER } [Re^{-1}]} \Delta^* \cdot \vec{v}^*$$

IF THESE NUMBERS ARE SUFICENTLY LARGE THAN THE NON LINEAR PART & WILL BE DIMINISHED.

FLUID MECHANICS IS CLASSICAL THEORY, THERE ARE NO SCALES ASSOCIATED WITH IT.

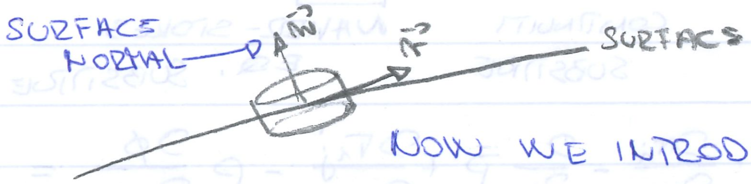
$$ST = \frac{L}{TV} \; ; \; E_M = \frac{P}{\rho V^2} \; ; \; \overbrace{F_r = \frac{V}{\sqrt{G}} \; ; \; R_e = \frac{\eta}{\rho V L}}^{\text{CAREFULL ABOUT}}$$

THESE, THEY BOTH ARE INVERSE.

2 FLOWS CAN BE DESCRIBED WITH THE SAME SET OF SCALING NUMBERS, THEY ARE SCALLED VERSIONS OF EACH OTHER BECAUSE THEY FOLLOW THE SAME DIMENSIONLESS NAVIER-STOKES-EQUATION.

A17 BOUNDARY CONDITION

(NOT VERY RELEVANT IN ASTRO PHYSICS)



NOW WE INTRODUCE INTEGRATION OVER THIS SELECTED VOLUME:

$v_\perp$: $\int dV \operatorname{div}\vec{v} = \int_{\partial V} d\vec{A}\,\vec{v} = \int_{\partial V} dA \cdot \underbrace{\vec{n}\cdot\vec{v}}_{0} = 0$ FOR INCOMPRESSIBLE FLUIDS

$\vec{v}\cdot\vec{m} = 0$ IS BECAUSE THE FLOW CANNOT PENETRATE THE SURFACE (NO FLOW IN OR OUT OF THE SURFACE).

$v_\parallel$: DEPEND ON A VISCOSITY

- VISCOSITY $\vec{v} = 0$ (THERE WILL BE SHEAR FORCE AND VISCOUS FORCES)
- NO-VISCOSITY, THEN $\vec{v}_\parallel$ IS UNCONSTRAINED

THIS APPLY FOR THE SURFACE, INSIDE THE FLOW THE $\vec{v}_\parallel$ IS AS WELL UNCONSTRAINED.

A18 CONSERVATION LAWS

- **CONSERVATION OF MASS** (CONTINUITY EQUATION)

$$\frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\vec{v}) = 0$$

IN COMPONENTS: $\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) = 0$

- **CONSERVATION OF MOMENTUM** (NAVIER-STOKES EQUATION)

THE DERIVATIVE OF MOMENTUM DENSITY:

$$\frac{\partial}{\partial t}(\rho\vec{v}) = \frac{\partial\rho}{\partial t}\vec{v} + \rho\frac{\partial\vec{v}}{\partial t}$$

IN COMPONENTS: $\frac{\partial}{\partial t}(\rho v_i) = \underbrace{\frac{\partial\rho}{\partial t}}_{\text{CONTINUITY SUBSTITUTE}} v_i + \rho\underbrace{\frac{\partial v_i}{\partial t}}_{\text{NAVIER-STOKES EQ. SUBSTITUTE}} =$

$$= -v_i\frac{\partial}{\partial x_j}(\rho v_j) - \rho v_j \cdot \frac{\partial v_i}{\partial x_j} - \frac{\partial}{\partial x_i}P + \frac{\partial\sigma'_{ij}}{\partial x_j} - \rho\frac{\partial\phi}{\partial x_i} =$$

$$= -\frac{\partial}{\partial x_j}\left[\rho v_i v_j + P\delta_{ij} - \sigma'_{ij}\right] - \rho\frac{\partial\phi}{\partial x_i} = \frac{\partial}{\partial t}(\rho v_i) \; *$$

WE DEFINE MOMENTUM FLUX TENSOR AS $\Pi_{ij} = \rho v_i v_j + P\delta_{ij} - \sigma'_{ij}$

FLUX OF THE i-TH COMPONENT INTO THE j-DIRECTION

NOTE ON MAGIC

USE $\frac{\partial}{\partial x_i}P = \frac{\partial P}{\partial x_j}\delta_{ij} = \frac{\partial}{\partial x_j}(P\delta_{ij})$

* • $\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = -\frac{\partial}{\partial x_i}\phi$ (IN COORDINATE FREE REPRES.)

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \mathrm{div}\,\Pi_{ij} = -\nabla\phi$$

CORRESPONDS TO A TRANSPORT EQUATION WITH $\nabla\phi$ AS A SOURCE.

• CONSERVATION OF (KINETIC) ENERGY DENSITY

LET'S IMAGINE: FLUID ELEMENT FLOWS AND CAN CHANGE IT'S VELOCITY (KINETIC ENERGY), BUT AS IT MOVES ALONG ITS MASS IS NOT GOING TO CHANGE. $M=\int dv\rho$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}\vec{v}^{2}\right)=\rho\vec{v}\frac{\partial\vec{v}}{\partial t}$$

IN COMPONENTS:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}v_i v_i\right)=\rho v_i\frac{\partial v_i}{\partial t}\Rightarrow$$

KINETIC ENERGY DENSITY $\frac{\rho}{2}\vec{v}^2$ OF A FLUID ELEMENT IN THE LAGRANGE FRAME, SIMILAR TO THE DERIVATION OF THE NAVIER STOKES EQUATION, WE ASSUME NO LOSS OF MATTER OF THE FLUID ELEMENT SO $\frac{d}{dt}\int_V dv\rho=0$. NOW WE SUBSTITUTE THE NAVIER-STOKES EQUATION FOR $\frac{\partial v_i}{\partial t}$:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}\vec{v}^{2}\right)=-\rho v_i v_j\frac{\partial v_i}{\partial x_j}-v_i\frac{\partial}{\partial x_i}p+v_i\frac{\partial}{\partial x_j}\sigma'_{ij}-\rho v_i\frac{\partial\phi}{\partial x_i}=$$

$$=-v_i\frac{\partial}{\partial x_i}\left(\rho\frac{\vec{v}^{2}}{2}\right)-v_i\frac{\partial}{\partial x_i}p+\frac{\partial}{\partial x_i}\left(v_i\sigma'_{ij}\right)-\sigma'_{ij}\frac{\partial v_i}{\partial x_j}-\rho v_i\frac{\partial\phi}{\partial x_i}$$

$$\frac{\partial}{\partial t}\left(\rho\vec{v}^{2}\right)=-\vec{v}\cdot\nabla\cdot\left(\frac{\rho v^2}{2}+p\right)+\nabla\cdot\left(\sigma'\cdot v\right)-\sigma'_{ij}\frac{\partial v_i}{\partial x_j}-\rho\vec{v}\nabla\phi$$

IDENTITY (PRODUCT RATE)

$$\nabla\cdot\left[\vec{v}\cdot\left(\frac{\rho}{2}\vec{v}^{2}+p\right)\right]=\left(\frac{\rho}{2}v^2+p\right)\operatorname{div}\vec{v}+\vec{v}\cdot\nabla\left(\frac{\rho v^2}{2}+p\right)$$

+ INCOMPRESSIBILITY $\operatorname{div}\vec{v}=0$ (MAKES SECOND TERM DISSAPPEAR)

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}\vec{v}^{2}\right)=-\nabla\cdot\left[\vec{v}\cdot\left(\frac{\rho v^2}{2}+p\right)-\sigma'\cdot\vec{v}\right]-\rho\vec{v}\nabla\phi-\sigma'_{ij}\frac{\partial v_i}{\partial x_j}$$

APPLY AN INTEGRATION OVER THE VOLUME OF A FLUID ELEMENT + GAUSS THEOREM

$$\frac{d}{dt}\int_V dV \frac{\rho}{2}\vec{v}^2 = -\int_{\partial V} d\vec{A}\cdot\vec{v}\cdot\frac{\rho\vec{v}^2}{2}$$ FLUX OF KIN ENERGY THROUGH SURFACE

$$-\int_{\partial V} d\vec{A}\cdot\vec{v}\cdot p$$ WORK DONE BY PREASSURE

$$+\int_{\partial V} d\vec{A}\cdot(\sigma'\cdot\vec{v})$$ WORK DONE BY SURFACE FORCES

$$-\int_V dV\cdot\rho\cdot\nabla\phi\cdot\vec{v}$$ WORK DONE BY EXTERNAL FORCES (LIKE GRAVITY)

$$-\int_V dV\,\sigma'_{ij}\boxed{\frac{\partial v_i}{\partial x_j}}$$ DISSIPATION OF ENERGY BY FRICTION

↳ VELOCITY TENSOR ϵ_{ij}

PLEASE BE CAREFUL VISCOSITY APPEARS TWICE

- SURFACE FORCES ACTING ON A FLUID ELEMENT
  $\int d\vec{A}(\sigma'\cdot\vec{v})$
- DISSIPATION INSIDE THE VOLUME → GENERATION OF HEAT
  $\int_V dV\sigma'_{ij}\frac{\partial v_i}{\partial x_j} = \int_V dV\, tr(\sigma'\cdot\epsilon)$ WITH SYM. VELOCITY TENSOR

$$\frac{\partial E}{\partial t} = -\frac{\eta}{2}\cdot tr(\epsilon^2)$$ WITH TERMAL ENERGY DENSITY.

A19 BERNOULLI EQUATION

- CONSERVATION OF KINETIC ENERGY FOR

① IDEAL FLUIDS (VISCOSITY = 0 (INVISCID) $\eta=\zeta=0$)

② INCOMPRESIBLE FLUIDS ($div\,\vec{v}=0$) STATIONARY FLOWS

$$\partial_t(\;)=0$$

$$\underbrace{\nabla\cdot\left[\vec{v}\left(\frac{\rho}{2}\vec{v}^{2}+p\right)\right]+\rho\vec{v}\nabla\phi}$$ THESE ARE THE RELEVANT TERMS

$$\vec{v}\cdot\nabla\cdot\left[\underbrace{\underbrace{\frac{\rho}{2}\vec{v}^{2}}_{\text{KIN. ENERGY}}+p+\phi}_{\text{TOTAL ENERGY}}\right]$$ WITH TOTAL ENERGY $E=\frac{\rho}{2}\vec{v}^{2}+p+\phi$

$\vec{v}\cdot\nabla E$ SUBSTITUTE INTO THE ENERGY CONSERVATION EQUATION

$$\frac{dE}{dt}=\vec{v}\cdot\nabla E$$

$\vec{v}\cdot\nabla$ ~ RATE OF CHANGE OF A QUANTITY WHEN DISPLACED ALONG A STREAM LINE, WHICH IS PARALLEL TO $\vec{v}$.

A20 RELATION TO RELATIVISTIC FLUID MECHANICS

LET'S CONSIDER DENSITY ρ

$\rho\to\gamma^{2}\cdot\rho$ ① LENGTH CONTRACTION - SAME DATA IN SMALLER VOLUME
② INCREASE IN MASS

ρ CAN'T BE SCALED, BUT MUST BE PART OF THE RELATIVISTIC ENERGY-MOMENTUM TENSOR $T^{00}\sim\rho\cdot c^{2}$

$$T^{\mu\nu}=\begin{pmatrix}\rho & & & 0\\ & p & & \\ & & p & \\ 0 & & & p\end{pmatrix}\xrightarrow[\text{TRANSF.}]{\text{LORENTZ}}\quad \overset{\text{BOOST}}{T^{\mu\nu}}\to\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}T^{\rho\sigma}$$

$$T^{\mu\nu}=\left(\rho+\frac{p}{c^{2}}\right)u^{\mu}u^{\nu}-p\cdot\eta^{\mu\nu}\leftarrow T^{\mu}\begin{pmatrix}\rho & & & 0\\ & p & & \\ & & \rho & \\ 0 & & & \rho\end{pmatrix}$$

~~CONSERVATION LAW~~ THE ENERGY-MOMENTUM-TENSOR OBEYS THE POYNTING LAW (CONSERVATION LAW): $\partial_\mu T^{\mu}=0$ AND CONTAINS THE REST MASS ENERGY $\rho \cdot c^2$ AS WELL AS THE ISOTROPIC PREASSURE P.

COMPUTE DIVERGENCE WITH EXPLICIT FORM OF T^{μ}: $\partial_\mu T^{\mu}=0$

$$\partial_\mu \left(\rho+\frac{P}{c^2}\right)\cdot u^\mu \cdot u^\nu + \left(\rho+\frac{P}{c^2}\right)\cdot\left[\partial_\mu u^\mu \cdot u^\nu + u^\mu \partial_\mu u^\nu\right] - \partial_\mu P\, \eta^{\mu\nu} = 0$$

TRICS:

$u^\nu u_\nu = c^2$ FOR THE 4-VELOCITY: $\partial_\mu u^\nu \cdot u_\nu + u^\nu \partial_\mu u_\nu = 0$

-> EACH TERM MUST BE ZERO INDIVIDUALLY:

$u_\nu (\partial_\mu T^{\mu\nu}) = 0$; $\partial_\mu (\rho u^\mu) + \frac{P}{c^2}\partial_\mu u^\mu = 0$ (CONTINUITY)

RESUBSTITUTE:

$$\left(\rho+\frac{P}{c^2}\right)\underbrace{\partial_\mu u^\nu \cdot u_\mu}_{\frac{\partial}{\partial t}\vec{v}+(\vec{v}\nabla)\vec{v}} + \underbrace{\left(\frac{u^\mu u^\nu}{c^2}-\eta^{\mu\nu}\right)}_{\text{PROJECTION OF } \partial_\mu P \text{ PERPENDICULAR TO } u^\mu}\underbrace{\partial_\mu P}_{\text{grad} p} = 0$$

RELATIVISTIC EULER EQUAT.

- NON-RELATIVISTIC LIMIT $\gamma \simeq 1$; $\underbrace{u^\mu}_{\text{FOUR VELOCITY}}(c; \overset{\text{THREE VELOCITY}}{\vec{v}})$; $P \ll \rho c^2$

CONTINUITY:

$\partial_\mu(\rho u^\mu)=0$ IF $P \ll \rho c^2$ $= \frac{\partial}{\partial t}\rho + \text{div}(\rho\vec{v}) = 0$

$\rho \partial_\mu u^\nu \cdot u_\mu + \left(\frac{u^\mu \cdot u^\nu}{c^2} - \eta^{\mu\nu}\right)\partial_\mu P = 0$ IF $P \ll \rho c^2$

t-COMPONENT: IDENTICALLY FULFILLED, $\partial_\mu u^0 = 0$

x-COMPONENTS: $\rho \partial_\mu u^i \cdot u^\mu = -\delta^{ij}\, \partial_j P$

$$\rho \cdot \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v}\right) = -\nabla p$$

- CLASSICAL FLUID MECHANICS IS THE NORM VR. LIMIT OF ENERGY-MOMENTUM CONSERVATION $\partial_\mu T^{\mu\nu}=0$ PROJECTED $\parallel$ AND $\perp$ TO THE DIRECTION OF u^μ

- RELATIVISTIC FLUID MECHANICS INCLUDING GRAVITY! PHYSICS IN GRAVITY REQUIRES THE DEFINITION OF A CONNECTION Γ AND COVARIANT DERIVATIVE ∇_μ WITH THE PROPERTY $\nabla_\mu g^\mu = 0$ g^μ IS THE METRIC, REPLACING THE η^μ OF SPECIAL RELATIVITY.

$\nabla_\beta v^\mu = \partial_\beta v^\mu - \Gamma^\mu_{\alpha\beta} v^\beta$ COVARIANTE DERIVATIVE

$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} - \Gamma^\mu_{\alpha\mu} T^{\alpha\nu} - \Gamma^\nu_{\mu\beta} T^{\mu\beta}$ FOR THE COVARIANT ENERGY CONSERVATION

IN A FREELY FALLING FRAME ONE RELEVES LOCALLY SPECIAL RELATIVITY:

$g \to \eta$; $\nabla_\mu \to \partial_\mu$ (AND $\partial g = 0$! IMPLIES $\Gamma = 0$)

$\to \nabla_\mu T^{\mu\nu} = 0$ WITH GRAVITY $\leftrightarrow$ $\partial_\mu T^{\mu\nu}$ WITHOUT GRAVITY

WEAK FIELDS: $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \left(1+\frac{2\phi}{c^2}\right) c^2 dt^2 - \left(1-\frac{2\phi}{c^2}\right) d\vec{x}^{\,2}$
IMPLY $\Gamma \sim \nabla\phi$, WHICH ADDS A $-\nabla\phi$ TO THE EULERS EQUATION

METRIC

$$g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\phi}{c^2} & & \\ & -\left(1-\frac{2\phi}{c^2}\right) & \\ & & \ddots \end{pmatrix} ; \quad \Gamma \sim \nabla\phi$$

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla\phi$$

A20 POTENTIAL FLOWS

- VELOCITY POTENTIAL φ ; VELOCITY FIELD $\vec{v} = \nabla\varphi$ FROM ELECTROSTATIC
UNDER COMPRESSIBILITY CONDITION $\text{div}\,\vec{v} = \nabla\vec{v} = \nabla\nabla\varphi = \Delta\varphi = 0$
$\text{rot}\,\vec{v} = 0$

$\vec{v}\cdot\vec{n} = 0 = \frac{\partial\varphi}{\partial n} = \nabla\varphi\cdot\vec{n}$ ON A BOUNDARY (WALL)

- UNIQUENESS OF THE VELOCITY POTENTIAL - GREEN THEOREM
2 SCALE FIELDS φ ; ψ → VECTOR $\vec{M} = \varphi\nabla\psi$
$\text{div}\,\vec{M} = \varphi\Delta\psi + \nabla\varphi\nabla\psi$

GAUSS THEOREM

$$\int_V dV\,\text{div}\,\vec{M} = \int_{\partial V} d\vec{A}\,\vec{M} = \int_{\partial V} dA\cdot\vec{n}\cdot\vec{M}$$

$$\int_V dV\,\text{div}\,\vec{M} = \int_V dV[\varphi\Delta\psi + \nabla\varphi\nabla\psi] = \int_{\partial V} dA\cdot\varphi\underbrace{\nabla\psi\cdot\vec{n}}_{=\frac{\partial\psi}{\partial n}} =$$

GREEN THEOREM

$$= \int_{\partial V} dA\cdot\varphi\cdot\frac{\partial\psi}{\partial n}$$

NOW WE CAN INTERCHANGE φ AND ψ AND SUBTRACT FOR BOUNDARY PROBLEM*

*

$$\int_V dV[\varphi\Delta\psi - \psi\Delta\varphi] = \int_{\partial V} dA\left[\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right]$$

NOW LET'S ASSUME THAT THERE ARE TWO VALID SOLUTIONS φ A ψ ; *
THIS IS A CONSEQUENCE OF INCOMPRESSIBILITY.

$u = \varphi - \psi$; $\Delta u = \Delta\varphi = \Delta\psi = 0$ (INCOMPRESSIBILITY)

$$\int_V dV[u\underbrace{\Delta u}_{0} - (\nabla u)^2] = \int_{\partial V} dA\cdot u\underbrace{\frac{\partial u}{\partial n}}_{0} = 0$$

BECAUSE OF THE BOUNDARY: $\frac{\partial u}{\partial n} = 0$ NO FLOW ACCROSS BOUNDARY
n - NORMAL OF THE SURFACE

$\nabla u = 0 \Rightarrow \psi = \varphi + \text{CONST.}$ AND THE CONSTANT VANISHES IN DIFFERENTIAL

VELOCITY FIELD $\nabla\psi = \nabla\varphi = \vec{v}$

- STREAM FUNCTION ψ (NO RELATION TO ψ FROM PREVIOUS PARAGRAPH)

HOW CAN WE DESCRIBE FLOWS JUST USING THESE φ & ψ POTENTIAL WITHOUT ACTUAL VELOCITY. WELL HOW ψ WORK:

IF $\text{div}\,\vec{v}=0$ → THAN THERE MUST BE A POTENTIAL $\vec{A}=(0;0;\psi)$ WITH $\vec{v}=\text{rot}\,\vec{A}$ $\quad v_x=\frac{\partial\psi}{\partial y}\,;\ v_y=-\frac{\partial\psi}{\partial x}$

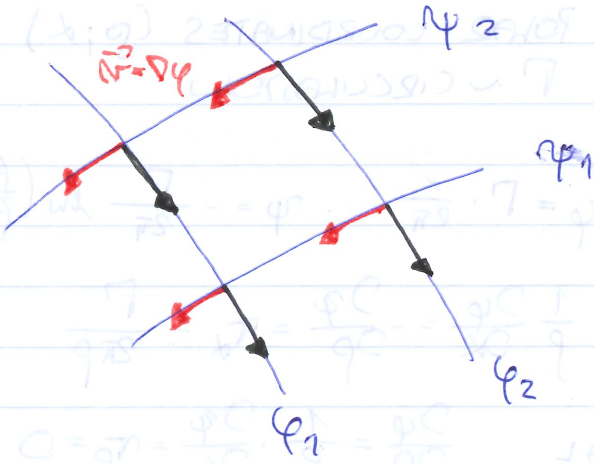
$\text{div}\,\vec{v}=0$ IS IDENTICALLY FULLFIELD FOR INCOMPRESSIBLE FLUIDS, BECAUSE $\text{div}\,\text{rot}\,\vec{A}=0=\text{div}\,\vec{v}$.

STREAM LINES: CURVES OF CONSTANT $\psi \rightarrow \vec{v}\perp\nabla\psi$ ~~PERPENDICULA~~

$$\vec{v}\cdot\nabla\psi = v_x\frac{\partial\psi}{\partial x}+v_y\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}=0$$

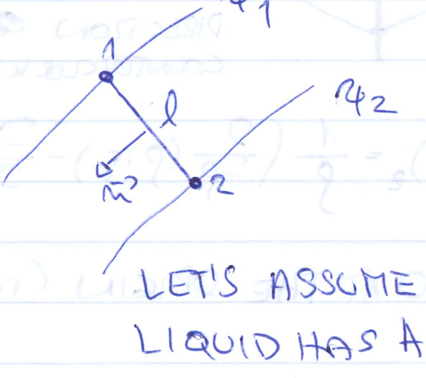
PERPENDICULAR

$$\rightarrow \vec{v}\cdot\nabla\psi=0\ ;\ \vec{v}\perp\nabla\psi\ ;\ \nabla\varphi\perp\nabla\psi$$



$$\nabla\varphi\cdot\nabla\psi=0$$

FLOW RATE $Q \sim \Delta\psi=\psi_2-\psi_1$



~~FLUX~~

Q - FLUX FROM SURFACE

$$Q=\int_1^2\vec{v}\cdot\vec{n}\,dl \qquad d\vec{l}=(dx;dy;0) \qquad \vec{n}\,dl=(dy;dx;0)$$

LET'S ASSUME THAT LIQUID HAS A CONST. DENSITY.

$$Q=\int_1^2\vec{v}\,\vec{n}\,dl=\int_1^2\left(\frac{\partial\psi}{\partial y}dy+\frac{\partial\psi}{\partial x}dx\right)=$$

$$=\int_1^2 d\psi=\psi_2-\psi_1=\underline{\overset{\text{DELTA}}{\Delta}\psi}$$

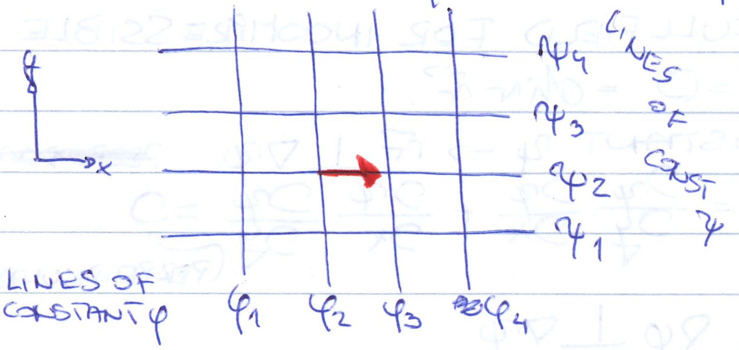
FLUX THROUGH THE SURFACE BETWEEN 2 LINES OF CONSTANT $\psi = \Delta\psi$.

- EXAMPLES OF VELOCITY POTENTIALS + STREAM FUNCTIONS

1) UNIFORM PARALLEL FLOW $\vec{v} = \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} v_x \\ 0 \end{pmatrix}$ IN x-DIRECTION

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} = v_x = u \qquad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} = v_y = 0$$

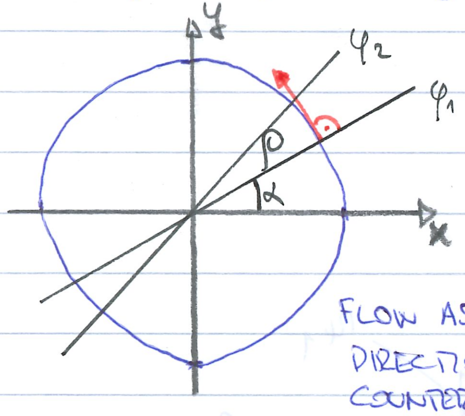
SOLVED BY $\varphi = ux$ AND $\psi = u\cdot y$



⇐ PRIMITIVES ONE TO ⇒
THE SUPERPOSITION YOU
CAN ASSEMBLE MANY
⇐ REALISTIC FLOWS

2) VORTEX FLOW → POLAR COORDINATES $(\rho ; \alpha)$

$v_\rho = 0$; $v_\alpha = \frac{\Gamma}{2\pi\rho}$ $\quad \Gamma \sim$ CIRCULATION



$$\varphi = \Gamma\cdot\frac{\alpha}{2\pi} \; ; \; \psi = -\frac{\Gamma}{2\pi}\ln\left(\frac{\rho_0}{\rho}\right)$$

$$\frac{1}{\rho}\frac{\partial\varphi}{\partial\alpha} = -\frac{\partial\psi}{\partial\rho} = v_\alpha = \frac{\Gamma}{2\pi\rho}$$

$$\frac{\partial\varphi}{\partial\rho} = \frac{1}{\rho}\cdot\frac{\partial\psi}{\partial\alpha} = v_\rho = 0$$

FLOW ASSIMUTAL DIRECTION ~~CLOCK~~ COUNTERCLOCKWISE

$$(\text{rot}\,\vec{v})_z = \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho v_\alpha) - \frac{\partial v_\rho}{\partial\alpha}\right) = \frac{1}{\rho}\left(v_\alpha + \rho\frac{\partial v_\alpha}{\partial\rho} - \frac{\partial v_\rho}{\partial\rho}\right) = 0$$

If $\rho \neq 0$ AT THE ORIGIN $(\text{rot}\,\vec{v})_z = \Gamma$ (BY APPLYING STOKES-THEOREM)

3) SOURCES AND SINKS - INTERCHANGED QUANTITIES COMPARED TO VORTEX FLOW

$v_\rho = \frac{Q}{2\pi\rho}$; $v_\alpha = 0$; Q ~ SOURCE STRENGTH

$$\varphi = \frac{Q}{2\pi}\cdot \ln\left(\frac{\rho}{\rho_0}\right) ; \quad \psi = Q\cdot\frac{\alpha}{2\pi}$$

$$\frac{1}{\rho}\frac{\partial\varphi}{\partial\alpha} = \frac{\partial\psi}{\partial\rho} = 0$$

$$\frac{\partial\varphi}{\partial\rho} = \frac{1}{\rho}\frac{\partial\psi}{\partial\alpha} = \frac{Q}{2\pi\rho}$$ RADIAL FLOW

$$\mathrm{div}\,\vec{v} = \frac{1}{\rho}\left(\frac{\partial(\rho v_\rho)}{\partial\rho} + \frac{\partial v_\alpha}{\partial\alpha}\right) = \frac{1}{\rho}\left(v_\alpha + \rho\cdot\frac{\partial v_\alpha}{\partial\rho} + \frac{\partial v_\rho}{\partial\alpha}\right) = 0$$

If $\rho \neq 0$ AT THE ORIGIN $\mathrm{div}\,\vec{v} = Q$ (BY APPLYING GAUSS-THEOR.)

A21 CONFORMAL MAPPINGS - COMPLEX DIFFERENTIABILITY FUNCTIONS

COMPLEX DIFFERENTIABILITY HAS 4 ASPECTS:

① COMPLEX DERIVATIVE $\frac{dg}{dz}$ EXISTS AND IS UNIQUE

② ANALYTIC: CAUCHY - RIEMANN DIFFERENTIAL EQUATIONS APPLY

③ REGULAR: LOOP INTEGRALS $\oint d\xi\, g(\xi) = 0$

④ HOLOMORPHIC $g(z) = \oint \frac{d\xi}{2\pi i}\cdot\frac{g(\xi)}{\xi - z}$

VALUES A LOOP DEFINE EVERY VALUE IN THE INTERIOR

- **HOLOMORPHISM**

LET'S WRITE $g(\xi) = g(z) + g(\xi) - g(z)$

$$g(z) \overset{?}{=} \frac{1}{2\pi i}\oint d\xi \frac{g(\xi)}{\xi - z} = \frac{1}{2\pi i}\oint d\xi\frac{g(z)}{\xi - z} + \frac{1}{2\pi i}\cdot\oint d\xi \frac{g(\xi)-g(z)}{\xi - z} =$$

$$= \frac{g(z)}{2\pi i}\underbrace{\oint d\xi\frac{1}{\xi - z}}_{2\pi i} + \frac{1}{2\pi i}\underbrace{\oint d\xi\frac{g(\xi)-g(z)}{\xi - z}}_{*} = g(z) = \longrightarrow$$

*

$$\le \oint d\xi \left(\frac{|g(\xi) - g(z)|}{\xi - z} \right) \le \max |g(\xi) - g(z)| \cdot \oint d\xi \frac{1}{\xi - z}$$

$= g(z)$ IF THE SECOND TERM IS $\le$ ~~ε~~ ε BY SELECTION OF~~A~~ THE INTEGRATION BOUNDARY.

THIS CAN BE EXTENDED BY TO n-FOLD DIFFERENTIATION

$$\frac{d^n}{d_t^{n}} g(z) = \frac{n!}{2\pi i} \oint d\xi \frac{g(\xi)}{(\xi - z)^n}$$

HOLOMORPHIC FUNCTIONS HAVE HOLOMORPHIC DERIVATIVES.

- COMPLEX DIFFERENTIABILITY

$$\left.\frac{dg}{dz}\right|_{\xi} = \lim_{z \to \xi} \frac{g(z) - g(\xi)}{z - \xi}$$

EXISTS AND IS UNIQUE (THE DETAILS OF $z - \xi$ DON'T MATTER)

- CAUCHY-RIEMANN DIFFERENTIAL EQUATIONS

$$\frac{dg}{dz} = \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z}$$

→ SHOULD BE INDEPENDENT ON Δz

$$\left.\begin{aligned} \frac{\partial g}{\partial x} &= \lim_{\Delta x \to 0} \frac{g(x + iy + \Delta x) - g(x + iy)}{\Delta x} \\ \frac{1}{i} \frac{\partial g}{\partial y} &= \lim_{\Delta y \to 0} \frac{g(x + iy + i\Delta y) - g(x + iy)}{i \Delta y} \end{aligned}\right\}$$

SHOULD BE EQUAL TO $\frac{dg}{dz}$

DECOMPOSE INTO REAL AND IMAGINARY PARTS $g(z) = u(x+iy) + iv(x+iy)$ BUT IF THE FUNCTION IS COMPLEX DIFFERENTIABLE, THE WAY IN WHICH THE DERIVATIVE IS COMPOSED SHOULD NOT MATTER.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{AND} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

THAN $$\frac{\partial g}{\partial x} = \frac{\partial (u+iv)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\shortparallel$$

$$\frac{1}{i}\frac{\partial g}{\partial y} = \frac{\partial (u+iv)}{\partial y} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \quad \text{WITH } \frac{1}{i} = -i$$

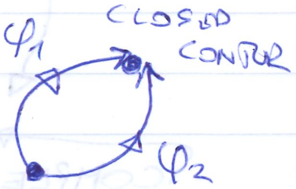
- REGULARITY CONDITION

CAUCHY THEOREM - COUNTUR INTEGRALS OF COMPLEX DIFFERENTIAL FUNCTIONS DON'T DEPPEND ON INTEGRATION PATH.

LOOP INTEGRALS - CLOSED CONTOUR, IF $z(a) = z(b)$

~~CAUCHY THEOREM~~

$$\int_{\varphi_1} dz\, g(z) + \int_{\varphi_2} dz\, g(z) = \oint dz\, g(z) = 0$$



REWRITE CAUCHY THEOREM

$$\oint dz\, g(z) = \oint (dx + i\,dy)(u + iv) = \oint (\underbrace{u\,dx - v\,dy}_{R}) + i\oint(\underbrace{u\,dy + v\,dx}_{I}) = \oint d\vec{r}\,\vec{R} + i\oint d\vec{r}\cdot\vec{I}$$

REINTERPRET REAL AND IMAGINARY PATHS AS SCALED PRODUCTS:

$\vec{R} = (u; -v; 0)$; $\vec{I} = (v; u; 0)$ WITH $d\vec{r} = (dx; dy; 0)$

THIS LEEDS TO

APPLY STOKES THEOREM - TRANSFORM LOOP INTEGRAL INTO SURFACE INTEGRAL

$$\oint_{\varphi = \partial A} d\vec{r}\cdot\vec{R} = \int_A d\vec{A}\,\text{rot}\,\vec{R} = \int_A dx\,dy\,[\underbrace{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}_{CR-1}] = 0$$

$$\oint_{\varphi=\partial A} d\vec{r}\cdot\vec{I} = \int_A d\vec{A}\,\mathrm{rot}\,\vec{I} = \int dx\,dy\,\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\text{CR-2}} = 0$$

- COMPLEX VELOCITY POTENTIAL

$$v_x = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{AND} \quad v_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

THIS CORRESPONDS TO THE CAUCHY-RIEMANN DIFFERENTIAL EQUAT.

if;

$$g(z) = \varphi(z) + i\psi(z) \; ; \; z = x + iy \sim \text{COMPLEX VELOCITY POTENTIAL}$$

$$w(z) = \frac{dg}{dz} = \begin{cases} \dfrac{\partial g}{\partial x} = \dfrac{\partial \varphi}{\partial x} + i\dfrac{\partial \psi}{\partial x} = v_x - i v_y \\[2ex] \dfrac{1}{i}\dfrac{\partial g}{\partial y} = \dfrac{\partial \psi}{\partial y} - \dfrac{i\,\partial \varphi}{\partial y} = v_x - i v_y \end{cases}$$

COMPLEX VELOCITY

- PHYSICAL INTERPRETATION OF LOOP INTEGRALS

$$C(z) = \oint_\xi dz\, w(z) = \oint_\xi (dx + i\,dy)(v_x + i v_y) =$$

$$= \oint_\xi (v_x dx + v_y dy) + i\oint (v_x dy - v_y dx) =$$

$$= \oint_\xi d\vec{l}\cdot\vec{v} \quad + i\oint_\xi dl\cdot\vec{v}\cdot\vec{n} = \Gamma + iQ$$

CIRCULATION SOURCE

- CONFORMAL MAPPINGS

- MAPPING OF COORDINATES $(x;y) \to (u;v)$ IS CALLED CONFORMAL IF THE MAPPING IS AN ANALITIC FUNCTION.

CONFORMAL MAPPINGS CONSERVE THE HARMONIC PROPERTY $\Delta_{xy} g = 0$ IF $(x;y) \to (u;v)$ WITH A NEW $G(u;v)$, THEN $\Delta_{uv} G = 0$.

$$\Delta_{xy} g = 0 \xrightarrow{(x;y)\to(u;v)} \Delta_{uv} G = 0$$

PHYSICAL CONSEQUENCE, ANY SOLUTION g OF A 2D INCOMPRESSIBLE FLOW CAN BE MAPPED INTO A MORE COMPLICATED GEOMETRY.

$$G(u;v) \to g(x;y) = G(u(x;y); v(x;y))$$

$$\frac{\partial g}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial G}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial G}{\partial v}$$

$$\frac{\partial^2 g}{\partial x^2} = \left(\frac{\partial u}{\partial x}\right)^2 \cdot \frac{\partial^2 G}{\partial u^2} + 2 \cdot \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \cdot \frac{\partial^2 G}{\partial u \partial v} + \left(\frac{\partial v}{\partial x}\right)^2 \cdot \frac{\partial^2 G}{\partial v^2} +$$

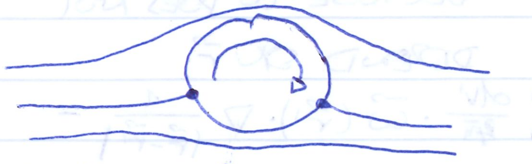
$$+ \frac{\partial^2 u}{\partial x^2}\frac{\partial G}{\partial x} + \frac{\partial^2 v}{\partial x^2}\frac{\partial G}{\partial v}$$

$$\frac{\partial g}{\partial y} = \dots \quad ; \frac{\partial^2 g}{\partial y^2}$$

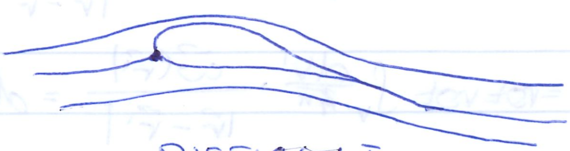
$$\overset{\text{LAPLACE}}{\Delta} g = \underbrace{\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}}_{0} = \dots = \underbrace{\left(\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}\right)}_{0} \cdot \underbrace{\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right.}_{\text{CORRATION FACTOR}}{}^2$$

$\Delta g \propto \Delta G$ WITH A POSITION DEPENDENT FACTOR

FLOW ACCROSS AIRPLANE WING



EASY: 1 SINK 1 SOURCE
1 VORTEX
1 UNIFORM FLOW



DIFFICULT GEOMETRY

A22 VORTICITY - GENERAL PROPERTIES

- VORTICITY + CIRCULATION

VELOCITY TENSOR $\frac{\partial v_i}{\partial x_j}$ ↗ SYMMETRY PART $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

↘ ANTISYM. PART $\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$

ω_{ij} IS TRACELESS + ANTISYMMETRIC

$\omega_i = 2\,\epsilon_{ijk}\,\omega_{jk} \to \vec{\omega} = \text{rot}\,\vec{v}$ (AXIAL VECTOR)

FOCUS ON INCOMPRESSIBLE FLUIDS $\text{div}\,\vec{v} = 0 \to$
ANALOGY TO ELECTRODYNAMICS $\vec{B} \sim \vec{v}$; $\vec{\omega} \sim \vec{j}$
+EXISTENCE OF A VECTOR POTENTIAL ~ STREAM FUNCTION ψ IN 2D

CIRCULATION

$$\Gamma = \int_{\partial A} d\vec{r} \cdot \vec{v} = \int_A d\vec{A} \cdot \text{rot}\,\vec{v} = \int_A d\vec{A} \cdot \vec{\omega} \quad \text{WITH STOKES THEOREM}$$

$$\to \int_{\partial A} d\vec{r} \cdot \vec{B} = \int_A d\vec{A} \cdot \text{rot}\,\vec{B} = \int_A d\vec{A}\,\vec{j} \quad \text{ENCLOSED EL. CURRENT}$$

- BIOT - SAVART LAW: RECOVER $\vec{v}$ FROM $\vec{\omega}$

$$\vec{v}(\vec{r}) = \int_V \frac{dV'}{4\pi}\,\omega(\vec{r}\,') \times \underbrace{\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}_{= \nabla \frac{1}{|\vec{r}-\vec{r}\,'|} = -\nabla' \frac{1}{|\vec{r}-\vec{r}\,'|}}$$

$$\text{rot}\,\vec{v} = \text{rot} \int_V \frac{dV'}{4\pi}\,\vec{\omega}(\vec{r}\,') \times \nabla \frac{1}{|\vec{r}-\vec{r}\,'|} =$$

$$= \text{rot} \int_V \frac{dV'}{4\pi} \cdot \nabla \times \left(\frac{\vec{\omega}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) = \quad \text{BECAUSE } \vec{\omega} \text{ DOES NOT DEPEND ON } \vec{r}$$

$$= \text{rot}\,\text{rot} \int_V \frac{dV'}{4\pi} \cdot \frac{\vec{\omega}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \text{div} \int_V \frac{dV'}{4\pi} \cdot \vec{\omega}(\vec{r}\,') \cdot \nabla \frac{1}{|\vec{r}-\vec{r}\,'|} -$$

$$- \int_V \frac{dV'}{4\pi}\,\vec{\omega}(\vec{r}\,') \cdot \Delta \frac{1}{|\vec{r}-\vec{r}\,'|} = *$$

$$= \text{div} \int_V \frac{dV'}{4\pi} \underbrace{\nabla'\vec{\omega}(\vec{r}\,')}_{=0} \cdot \frac{1}{|\vec{r}-\vec{r}\,'|} + \int_V dV'\,\vec{\omega}(\vec{r}\,') \cdot 4\pi\,\delta_D(\vec{r}-\vec{r}\,') = \vec{\omega}$$

A23 VORTICITY EQUATION -> TIME EVOLUTION OF $\vec{\omega}$

STARTING POINT NAVIER-STOKES EQUATION FOR INCOMPRESSIBLE FLUIDS

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla\phi + \mu\Delta\vec{v}$$

USE $(\vec{v}\nabla)\vec{v} = \nabla\frac{\vec{v}^2}{2} - \vec{v}\times\nabla\times\vec{v}$, WITH $\vec{\omega} = \text{rot}\,\vec{v}$

$$\vec{v}\times\nabla\times\vec{v} = \vec{v}\times\vec{\omega} = -\vec{\omega}\times\vec{v} = -(\nabla\times\vec{v})\times\vec{v} = -\frac{\nabla\vec{v}^2}{2} + (\vec{v}\cdot\nabla)\vec{v}$$

BUILD ROTATION OF THE NAVIER-STOKES-EQ

$$\frac{\partial\vec{v}}{\partial t} + \nabla\frac{\vec{v}^2}{2} - \vec{v}\times\vec{\omega} = -\frac{\nabla p}{\rho} - \nabla\phi + \mu\Delta\vec{v} \quad /\text{rot}$$

$$\frac{\partial\,\text{rot}\,\vec{v}}{\partial t} - \text{rot}(\vec{v}\times\vec{\omega}) = \mu\,\text{rot}(\Delta\vec{v})$$

- $\text{rot}\,\nabla(\vec{v}^2) = 0$ } ROTATION OF A GRADIENT
- $\text{rot}\,\nabla\phi = 0$
- IF $p = p(\rho)$, AND NOTHING ELSE

$$\nabla p = \frac{\partial p}{\partial\rho}\cdot\nabla\rho \longrightarrow \nabla p\times\nabla\rho = 0 \,!$$

$$\text{rot}\left(\frac{\nabla p}{\rho}\right) = \frac{\text{rot}\,\nabla p}{\rho} - \frac{\nabla p\times\nabla\rho}{\rho^2} = -\frac{\nabla p\times\nabla\rho}{\rho^2} = 0$$

"BARIONIC TERM" ONLY ACTIVE FOR EOS $p \neq p(\rho)$

SIMPLIFY THE TWO REMAINING TERMS

- $\text{rot}(\vec{A}\times\vec{B}) = \vec{A}\cdot\text{div}\,\vec{B} - \vec{B}\cdot\text{div}\,\vec{A} + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$

$$\rightarrow \text{rot}(\vec{v}\times\vec{\omega}) = (\vec{\omega}\cdot\nabla)\vec{v} - \underbrace{(\vec{v}\cdot\nabla)\vec{\omega}}_{\text{ADVECTIVE}} + \vec{\omega}\cdot\underbrace{\text{div}\,\vec{v}}_{\substack{=0\\ \text{INCOMPRESSIBLE}}} + \vec{v}\cdot\underbrace{\text{div}\,\vec{\omega}}_{\substack{=0\\ \text{DIVERGENCE OF rot}}} =$$

$$= (\vec{\omega}\cdot\nabla)\vec{v} - (\vec{v}\nabla)\vec{\omega}$$

- $\text{rot rot}(\vec v) = \nabla \,\text{div}\, \vec v - \Delta \vec v$

$$\rightarrow (\Delta \vec v) = \text{rot}(\underbrace{\nabla\cdot\text{div}\,\vec v}_{=0} - \text{rot rot}\,\vec v) = - \text{rot rot rot}\,\vec v =$$

$$= - \text{rot rot}\,\vec\omega = \Delta\omega - \nabla\,\underbrace{\text{div}\,\vec\omega}_{=0} = \Delta\vec\omega$$

$\rightarrow \text{rot}(\Delta\vec v) = \Delta\vec\omega$ · IF $\text{div}\,\vec v = 0$

LET'S COLLECT ALL RESULTS FOR THE VORTICITY EQUATION

$$\underbrace{\frac{\partial}{\partial t}\vec\omega + (\vec v\nabla)\vec\omega}_{\text{ADVECTIVE DERIVATIVE}} = (\vec\omega\cdot\nabla)\vec v + \underbrace{\mu\Delta\vec\omega}_{\text{VISCOSITY}}$$

ADVECTION: GIVEN BY $\vec v = \int \frac{d^3r}{4\pi}\,\vec\omega\times\nabla\frac{1}{|\vec r - \vec r'|}$ ~ NONLOCAL

VISCOSITY GENERATES DIFFUSION: $\frac{\partial\vec\omega}{\partial t} - \mu\Delta\vec\omega = 0$

↳ GREEN FUNCTION $G(r,t) = \frac{1}{(2\pi r)^{2/3}} \cdot e^{\left(-\frac{r^2}{2\mu t}\right)}$

DIFFUSION $\propto \sqrt{t}$ ~ RANDOM WALK

VORTICITY EQUATION IS NONLINEAR $(\vec v\nabla)\omega$ AND $(\vec\omega\cdot\nabla)\vec v$ + NONLOCAL

GENERATION ONLY DUE TO BARIONIC TERM $\nabla p \times \nabla\rho$

GRAVITY CAN'T DO IT: $\text{rot}\nabla\Phi = 0$ (WHY DO GALAXIES ROTATE)

A24 ENSTROPHY AND PALINSTROPHY

- ENSTROPHY $e = \frac{\vec\omega^2}{2}$

$$\frac{\partial}{\partial t}e = \frac{1}{2}\frac{\partial}{\partial t}\vec\omega^2 = \frac{1}{2}\tilde\omega\frac{\partial\vec\omega}{\partial t} = \frac{\vec\omega}{2}\left((\vec v\cdot\nabla)\vec\omega\right) = \sum_{ij}\omega_i\cdot\frac{\partial v_i}{\partial x_j} \sim O(\omega^3)$$

$$\rightarrow \frac{\partial}{\partial t}\omega^2 \propto \omega^3$$

SOLVED BY $\omega \sim \dfrac{1}{\frac{1}{\omega_0} - t}$ DIVERGIS WHEN $t\omega_0 \to 1$

ω_0 - INITIAL VALUE

PALINSTROPHY $\vec{p} = \text{rot}\, \vec{\omega} = \text{rot}\,\text{rot}\, \vec{v} = \nabla \text{div}\, \vec{v} - \Delta \vec{v}$

$\vec{p} = -\Delta \vec{v}$ FOR INCOMPRESSIBLE FLUIDS

(ONLY NEEDED FOR IMPRESSING PEOPLE)

A25 DYNAMICS OF THE CIRCULATION → KELVIN-THEOREM

(IN THE GERMAN LITERATURE, THEY ARE CALLED HELMHOLTZ-TH.)

$\Gamma = \int_{\partial A} d\vec{r} \times \vec{v} = \int_A d\vec{A} \cdot \vec{\omega}$ STOKES-THEOREM WITH $\vec{\omega} = \text{rot}\, \vec{v}$

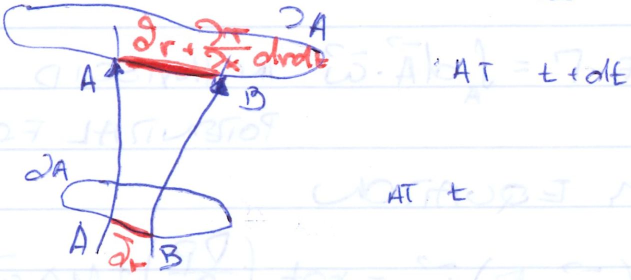
ASSUME $\begin{cases} \text{IDEAL FLUIDS, VISCOSITY } \mu = 0 \\ \text{POTENTIAL FORCE } \nabla \phi \text{, NO OTHER FORCES} \\ \text{CONSTANT DENSITY (OR AT MOST A FUNCTION OF } p) \end{cases}$

→ IDENTICAL TO THE ASSUMPTIONS OF THE BERNOULLI-THEOR

KELVIN-THEOREM: THE ADVECTED CIRCULATION IS CONSERVED

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \int_A d\vec{r} \times \vec{v} = \frac{d}{dt} \int_A d\vec{A} \cdot \vec{\omega} = 0$$

WITH ADVECTIVE DERIVATIVES $\dfrac{d}{dt}(\;) = \dfrac{\partial}{\partial t} + \vec{v} \cdot \nabla (\ldots)$



$$\frac{d}{dt} \int_{\partial A} \partial \vec{r} \cdot \vec{v} = \int_{\partial A} \frac{d}{dt} \partial \vec{r}\, \vec{v} + \int_{\partial A} \partial \vec{r} \cdot \frac{d\vec{v}}{dt}$$

$$\text{WITH } \int_{\partial A} \frac{d}{dt} \partial \vec{r}\, \vec{v} = \int_{\partial A} \frac{d\, \partial r_i}{dt} v_i = \int_{\partial A} \frac{\partial v_i}{\partial x_j} \partial r_j \cdot v_i =$$

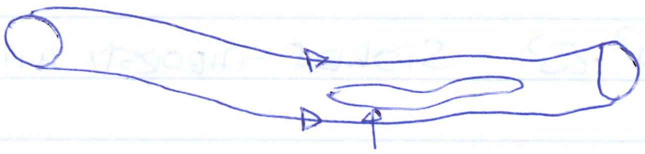
$$= \int_{\partial A} \partial r_j \cdot \frac{\partial}{\partial x_j} \left(\frac{v_i^2}{2} \right) = \int_{\partial A} \partial \vec{r} \cdot \nabla \left(\frac{\vec{v}^2}{2} \right)$$

AND $\int_{\partial A} d\vec{r} \cdot \frac{d\vec{v}}{dt} = \int_{\partial A} d\vec{r} \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} \right) = \int_{\partial A} d\vec{r} \cdot \left(-\frac{\nabla p}{\rho} - \nabla\phi \right)$

LOOK AT BOTH TERMS: THEY ARE GRADIENTS OF SCALES, SO THEIR CORRESPONDING CIRCULATION NEEDS TO BE ZERO!

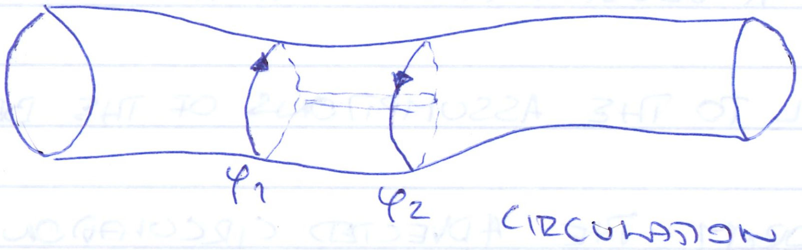
A26 CONCLUSIONS FROM THE KELVIN-THEOREM

- IF THE CIRCULATION $\Gamma = 0$, IT REMAINS ZERO $\rightarrow \vec{\omega} = 0$
- VORTICITY FOLLOWS THE FLOW BY ADVECTION



∂A ON SURFACE NO FLUX OF $\vec{\omega}$ OUT OR IN

- CIRCULATION IS CONSERVED ALONG A FLOW



CIRCULATION THROUGH φ_1 AND φ_2 ARE OPPOSITE AND EQUAL
$\rightarrow$ CONTRACT LOOP

A27 GENERATION OF VORTICITY

CONSERVATION OF $\Gamma = \int_A d\vec{A} \cdot \vec{\omega}$ IN INVISCID FLOWS WITH POTENTIAL FORCES

FULL VORTICITY EQUATION

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v}\nabla)\vec{\omega} = (\vec{\omega} \cdot \nabla)\vec{\omega} - \mathrm{rot}\left(\frac{\nabla P}{\rho}\right) + \mu \Delta \vec{\omega}$$

BAROMETRIC TERM

$$\mathrm{rot}\left(\frac{\nabla P}{\rho}\right) = -\frac{\nabla P \times \nabla \rho}{\rho} \neq 0 \quad \text{if } P = P(\rho; T)$$

, SUCH THAT PREASSURE AND DENSITY GRADIENTS ARE NOT EQUAL

NON-POTENTIAL FORCES

$\to$ CORIOLIS FORCE

$\to$ LORENTZ FORCE

- CORIOLIS FORCE $\to 2\vec{\Omega}\times\vec{v}$ APPEARS AS A TERM IN THE NS-EQ

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\nabla)\vec{v}=-\frac{\nabla p}{\rho}-\nabla\phi+\mu\Delta\vec{v}-2\vec{\Omega}\times\vec{v}$$

(SIMILARLY TO ϕ, AS. AN INERTIAL (GRAVITATIONAL FORCE)

$$\mathrm{rot}(\vec{\Omega}\times\vec{v})=\underbrace{(\vec{v}\cdot\nabla)\vec{\Omega}}_{\substack{=0\\ \text{if } \vec{\Omega}=\text{KONST}}}-(\vec{\Omega}\cdot\nabla)\vec{v}-\underbrace{\vec{\Omega}\,\mathrm{div}\,\vec{v}}_{\substack{=0\\ \text{INCOMPRESSIBLE}}}+\underbrace{\vec{v}\cdot\mathrm{div}\,\vec{\Omega}}_{\substack{=0\\ \text{if } \Omega=\text{KONST}}}$$

$$\to \mathrm{rot}(\vec{\Omega}\times\vec{v})=(\vec{\Omega}\cdot\nabla)\vec{v}$$

$$\frac{\partial}{\partial t}\vec{\omega}+(\vec{v}\nabla)\vec{\omega}=+\frac{\nabla p\times\nabla\rho}{\rho^2}+\mu\Delta\vec{\omega}-2(\vec{\Omega}\cdot\nabla)\vec{v}$$

IMPOTANCE: QUANTIFY WITH A DIMENSIONLESS NUMBER

$$\left|\frac{(\vec{v}\nabla)\vec{v}}{\vec{\Omega}\times\vec{v}}\right|\sim\frac{v^2/L}{\Omega\cdot v}=\frac{v}{\Omega L}=R_0 \quad \text{ROSSBY NUMBER}$$

- LORENTZ FORCE (NON-RELATIVISTIC LIMIT)

$$\vec{f}=\frac{q}{\rho}(\vec{E}+\vec{v}\times\vec{B}),\quad \mathrm{rot}\,\vec{f}=\frac{q}{\rho}\cdot\mathrm{rot}(\vec{v}\times\vec{B})=-\frac{q}{\rho}\cdot(\vec{B}\cdot\nabla)\vec{v}$$

IMPORTANCE

$$\left|\frac{(\vec{v}\cdot\nabla)\vec{v}}{q\cdot(\vec{B}\nabla)\vec{v}}\rho\right|=\frac{v\cdot\rho}{q\cdot B\cdot L}=\frac{1}{Re\cdot HA} \quad \text{HARTMANN-NUMBER}$$

WE'll GO THROUGH PLASMA PHYSICS IN DETAIL)

A28 STOKES EQUATION - FLOW AT LOW REYNOLDS NUMBER

- VORTICITY IMPORTANT IN TURBULENT FLOWS

$(\vec{v}\cdot\nabla)\vec{v}$ DOES NOT PLAY AN IMPORTANT ROLE $\to$ LINEAR THEORY (FLUID)

VISCOSITY DOMINATES

WHAT IS LEFT IN NAVIER-STOKES EQ:

$$-\frac{\nabla p}{\rho} - \nabla\phi + \mu\Delta\vec{v} = 0$$

UNDER CONDITION, THAT THE FLOW IS STATIC $\frac{\partial\vec{v}}{\partial t} = 0$ HONEY ON A TOAST

TWO GRADIENTS

$$\nabla p' = \nabla p + \rho\nabla\phi \longrightarrow \nabla p' = \mu\Delta\vec{v}$$

ABSORBING $\nabla\phi$ INTO ∇p

EQUIVALENT REPRESENTATION OF STOKE'S EQUATION

- $$\frac{\partial\sigma'_{ij}}{\partial x_i} = \eta\cdot\Delta v_j = \mu\rho\Delta v_i$$

  σ'_{ij} - STRESS TENSOR WHERE TRACE OF THIS TENSOR IS EQUAL TO PREASSURE p

  $$\longrightarrow \frac{\partial\sigma_{ij}}{\partial x_j} = 0 \quad \text{WITH } \sigma_{ij} = \sigma'_{ij} - p\delta_{ij}$$

  STRESS TENSOR DIVERGENS IS EQUAL TO ZERO

- $$\text{rot}\,\underbrace{\text{rot}\,\vec{v}}_{\vec{\omega}} = \nabla\underbrace{\text{div}\,\vec{v}}_{=0} - \Delta\vec{v} \longrightarrow -\frac{\nabla p}{\rho} = \mu\,\text{rot}\,\vec{\omega}$$

  INCOMPRESSIBLE FLUIDS

- $$\text{rot}\,\text{rot}\,\vec{\omega} = \nabla\underbrace{\text{div}\,\vec{\omega}}_{0} - \Delta\vec{\omega} \longrightarrow -\frac{1}{\mu}\text{rot}\,\nabla p = 0 = \Delta\vec{\omega}$$

  LAPLACE TYPE POTENTIAL PROBLEM

  FOR CONSTANT ρ

A3) SOLUTIONS TO THE STOKES EQUATION

- UNIQUENESS - ASSUME THAT THERE ARE 2 SOLUTIONS, IDENTICAL BOUNDARY CONDITIONS

$$\frac{\partial v_i}{\partial x_j} \overset{?}{=} \frac{\partial v'_i}{\partial x_j} \longrightarrow \text{IDENTICAL VELOCITY TENSORS?}$$

EQUALITY OF $\vec{v}$ THEN FOLLOWS FROM INTEGRATION WITH FIXED BOUNDARY CONDITION

$$\int_V dV \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i'}{\partial x_j}\right)^2 = \underbrace{\int_V dV \frac{\partial}{\partial x_j}\left[(v_i - v_i')\cdot\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i'}{\partial x_j}\right)\right]}_{① = 0} -$$

$$- \underbrace{\int_V dV (v_i - v_i')(\Delta v_i - \Delta v_i')}_{②}$$

① SURFACE INTEGRAL = 0 (GAUSS THEOREM) DUE TO IDENTICAL BOUNDARY CONDITIONS

$$② = -\frac{1}{\eta}\int_V dV\, (v_i - v_i')\cdot\left(\frac{\partial p}{\partial x_i} - \frac{\partial p'}{\partial x_i}\right) =$$

$$= -\frac{1}{\eta}\int_V dV \underbrace{\frac{\partial}{\partial x_i}(v_i - v_i')(p - p')}_{0 = \text{GAUSS TH.}} + \frac{1}{\eta}\int_V dV$$

$$\underbrace{\left(\frac{\partial v_i}{\partial x_i} - \frac{\partial v_i'}{\partial x_i}\right)}_{= 0 \text{ INCOMPRESSIBLE}} \cdot (p_i - p_i') = 0$$

→ VELOCITY TENSORS ARE EQUAL! IF $\operatorname{div}\vec{v} = 0$

→ IDENTITY OF $\vec{v}$ FOLLOWS THEN FROM INTEGRATION

A31 ENERGY DISSIPATION IN A STOKES FLOW

$$E = \int_V dV\, \epsilon_{ij}' \underbrace{\frac{\partial v_i}{\partial x_j}}_{\text{VELOCITY TENSOR}} = \frac{\eta}{2}\int_V dV \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 = 2\eta \cdot \int_V dV \epsilon^2$$

ENERGY DISSIPATION → MINIMAL FOR STOKES FLOW & $\nabla p = \eta \Delta \vec{v}$

NOW ASSUME THAT THERE IS A STRAIN TENSOR ϵ', WHICH DISSIPATES MORE ENERGY THAN ϵ

WHERE ϵ IS THE SOLUTION TO THE STOKES EQUATION & ϵ & ϵ' IS BOTH INCOMPRESSIBLE:

$$2\eta\int_V dV \epsilon'^2 = \underbrace{2\eta \cdot \int_V dV \epsilon^2}_{①} + \underbrace{2\eta\int_V dV (\epsilon' - \epsilon)^2}_{②} +$$

$$\underbrace{+4\eta\cdot\int_V dV\,(\epsilon'-\epsilon)\,\epsilon}_{③}$$

INTEGRAL ② IS NECESSARY POSSITIVE → $(\ \ )^2$ → +

IF WE CAN SHOW THAT ③ IS ZERO, THAN THE DISSIPATION WITH e' IS MORE EFFECTIVE THAN WITH e.

OUR GOAL WILL BE TO SHOW THAT $\int_V dV\, e'^2 > \int_V dV\, e^2$ BY SHOWING THAT ③ ≥ 0

$$\int_V dV\,(\epsilon'-\epsilon)\,\epsilon = \frac{1}{2}\Big[\underbrace{\int_V dV\left(\frac{\partial v_i'}{\partial x_j}-\frac{\partial v_i}{\partial x_j}\right)\frac{\partial v_i}{\partial x_j}}_{Ⓐ} + \underbrace{\int_V dV\left(\frac{\partial v_i'}{\partial x_j}-\frac{\partial v_i}{\partial x_j}\right)\frac{\partial v_j}{\partial x_i}}_{Ⓑ}\Big]$$

$$Ⓐ = \underbrace{\int_V dV\,\frac{\partial}{\partial x_j}\left[(v_i'-v_i)\frac{\partial v_i}{\partial x_j}\right]}_{\text{SURFACE INTEGRAL} = 0} - \int_V dV\,(v_i - v_i')\frac{\partial^2 v_i}{\partial x_j^2} =$$

$$= \frac{1}{\eta}\int_V dV\,(v_i'-v_i)\cdot\frac{\partial p}{\partial x_i} = \frac{1}{\eta}\Big(\underbrace{\int_V dV\,\frac{\partial}{\partial x_i}\left((v_i'-v_i)\,p\right)}_{0 = \text{GAUSS. TH.}} -$$

$$\underbrace{-\int_V dV\,\frac{\partial}{\partial x_i}(v_i'-v_i)\cdot p\Big)}_{0 \text{ INCOMPRESSIBLE}} = 0$$

$$Ⓑ = \int_V dV\left(\frac{\partial v_i'}{\partial x_j}-\frac{\partial v_i}{\partial x_j}\right)\cdot\frac{\partial v_j}{\partial x_i} = \int_V dV\,\frac{\partial}{\partial x_i}\underbrace{\left[(v_i - v_i')\cdot\frac{\partial v_j}{\partial x_i}\right]}_{=0 \text{ SURFACE}} -$$

$$\underbrace{-\int_V dV\,(v_i'-v_i)\underbrace{\frac{\partial^2}{\partial x_i\,\partial x_j}v_j}_{\frac{\partial}{\partial x_i}\operatorname{div}\vec{v} = 0}}_{\text{INCOMPRES} = 0} = 0$$

A32 REYNOLDS EQUATION

ASPECTS OF TURBULENCE → DYNAMICS $|(\vec{v}\cdot\nabla)\vec{v}| >> |\mu\Delta\vec{v}|$ (HIGH REYNOLDS NUMBER

→ CORRELATION + STATISTICS

→ SPECTRAL PROPERTIES

PROPERTIES OF TURBULENT FLOW:
- MANY TIMESCALES AND LENGTH SCALES
- UNORDERED "RANDOM" STRUCTURE
- NONLINEAR $(\vec{v}\nabla)\vec{v}$ IS DOMINANT, HIGH Re
- STRONG DEPENDENCE ON INITIAL CONDITIONS.

LET'S INTRODUCE AN IDEA ABOUT AVERAGING SCALE AND SEPARATE MACROSCOPIC FROM MICROSCOPIC MOTION ⟶ MACROSCOPIC FLOW PATH

↳ MICROSCOPIC FLUCTUATIONS

AVERAGE OF THE VELOC. FLUCTUATION = 0

$$\vec{v} = \langle\vec{v}\rangle + \vec{v}' \; ; \; p = \langle p\rangle + p' \quad \text{WITH} \quad \langle\vec{v}'\rangle = \langle p'\rangle = 0$$

AVERAGE OF THE PRESSURE FLUCTU. = 0

WHERE $\langle\ldots\rangle$ ARE ENSEMBLE AVERAGES: SUBSTITUTE INTO NAVIER-STOKE EQUATION

$$\rightarrow \rho\cdot\frac{\partial}{\partial t}\langle v_i\rangle + \rho\frac{\partial}{\partial x_j}\langle v_i v_j\rangle = \rho\frac{\partial}{\partial x_i}\phi - \frac{\partial}{\partial x_i}p + \eta\frac{\partial}{\partial x_j}\cdot$$

$$\cdot\left(\frac{\partial\langle v_i\rangle}{\partial x_j} + \frac{\partial\langle v_j\rangle}{\partial x_i} - \frac{2}{3}\frac{\partial}{\partial x_k}\langle v_k\rangle\cdot\delta_{ij}\right) + \overbrace{\rho\frac{\partial}{\partial x_j}\underbrace{\langle v_i' v_j'\rangle}_{\tau_{ij}}}^{\text{THIS LOOKS LIKE KIN. EN. DENSIT}}$$

WE DEFINE REYNOLDS TENSOR AS $\tau_{ij} = \langle v_i' v_j'\rangle$ IT IS SORT OF A VARIACE OF COMPONENTS OF THE VELOCITY (ALL THE FLUCTUATIONS OF THE VELOCITY), MACROSCOPIC MOTION DEPENDS ON MICROSCOPIC VELOCITY VARIANCE.

• $\mathrm{tr}\,\tau = \sum_i \tau_{ii} = \langle v_i' v_i' \rangle$ ~ TURBULEN PREASSURE

A33 BOUSSINESQ MODEL – PHENOMENOLOGICAL MODEL FOR τ_{ij}

TOHLECO JE NA PREDCHOZÍ STRANE

$$\rho\cdot\tau_{ij} = -\mu_t\cdot\left(\frac{\partial}{\partial x_j}\langle v_i\rangle + \frac{\partial}{\partial x_i}\langle v_j\rangle - \frac{2}{3}\frac{\partial}{\partial x_k}\langle v_k\rangle\delta_{ij}\right) +$$

$$+ \frac{2}{3}\rho\cdot k\cdot\delta_{ij}$$

KONST.

WITH $k = \frac{1}{2}\mathrm{tr}\langle v_i' v_j' \rangle$ AND FIXED μ_t

μ_t & k THEY SHOULD SOMEHOW MODEL HOW THE FLOW MACROSCOPICALY BEHAVE.

A34 CORRELATION FUNCTIONS OF THE VELOCITY FIELD

IDEA: FULLY DEVELOPED TURBULANCE IS CHARACTERISED BY RANDOM FIELDS, BUT THERE WILL BE CORRELATION IN THE DENSITY FIELD

PROBABILITY DISTRIBUTION FOR VELOCITY AT THE POINT $\vec{x}$ –

$$P(\vec{v}(\vec{x}); \vec{v}(\vec{y})) \neq p(\vec{v}(\vec{x}))\cdot p(\vec{v}(\vec{y}))$$

FOR THE VELOCITY FIELD $\vec{v}(\vec{x})$

IF THERE ARE CORRELATIONS ASSUME GAUSSIAN STATISTICS (EVEN THOUGH THEY ARE NOT TRUE IN THIS CASE)

$$p(\vec{v}(x); \vec{v}(y)) = \underbrace{\frac{1}{\sqrt{(2\pi)^6\cdot \det(C)}}}_{\text{NORMALIZATION FACTOR}} e^{\left(-\frac{1}{2}(\vec{v}(x); \vec{v}(y))\cdot C^{-1}\left(\frac{\vec{v}(x)}{\vec{v}(y)}\right)\right)}$$

C - MATRIX

$(\vec{v}(x); \vec{v}(y))$ – FORMS SIX DIMENSIONAL VECTOR

COVARIACE MATRIX

$$C = \begin{pmatrix} \underbrace{\langle \vec{v}(x) \otimes \vec{v}(x) \rangle}_{} & \langle \vec{v}(x) \otimes \vec{v}(y) \rangle \\ \underbrace{\langle \vec{v}(x) \otimes \vec{v}(y) \rangle}_{3 \times 3 \text{ MATRIX}} & \langle \vec{v}(y) \otimes \vec{v}(y) \rangle \end{pmatrix}$$

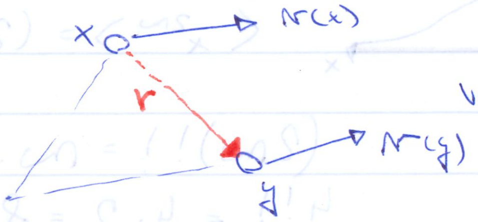
LETS DEFINE VELOCITY CORELATION FUNCTION:

$Q = \langle \vec{v}(y) \otimes \vec{v}(x) \rangle$; $Q_{ij} = \langle v_i(x) \cdot v_j(y) \rangle$

$Q_{ij}(\vec{r}) = Q_{ij}(-\vec{r})$ FOR ISOTROPIC TURBULENCE

$Q_{ij}(\vec{r}) = Q_{ij}(\vec{r})$ HOMOGENEUS TURBULENCE

DESCRIBES HOW i-COMPONENT OF THE VELOCITY IS RELATED TO THE j-COMPONENT



VELOCITY CORELATION FUNCTION

$\hookrightarrow Q_{ij} = \langle v_i(x) \cdot v_j(y) \rangle = \langle v_i(x) \cdot v_j(x+r) \rangle$

PROPERTIES OF THE VELOCITY CORELATION:

① $\frac{\rho}{2} Q_{ii}(0) = \frac{\rho}{2} \langle \vec{v}^2 \rangle$ ~ KINETIC ENERGY DENSITY

② $|Q_{ij}(r)| \leq |Q_{ij}(0)|$ CAUCHY-SWARZ INEQUALITY

③ $\rho \cdot Q_{ij}(0) = \epsilon_{ij}$ REYNOLDS STRESS

④ $\frac{\partial}{\partial r_i} Q_{ij} = \frac{\partial}{\partial r_j} Q_{ij} = 0$ FOR INCOMPRESSIBLE FLUIDS

PARAMETRISATION OF Q

$\ell(t) = 1/2 \cdot \mathrm{tr}\, Q = 1/2 \cdot \langle \vec{v}(x) \cdot \vec{v}(y) \rangle$

$$\left.\begin{aligned} v^2 \cdot f &= Q_{xx}(r \cdot \vec{e}_x) \\ v^2 \cdot g &= Q_{yy}(r \cdot \vec{e}_x) \end{aligned}\right\} \text{ WITH } v^2 = \sqrt{1/3 \langle \vec{v}^2 \rangle}$$

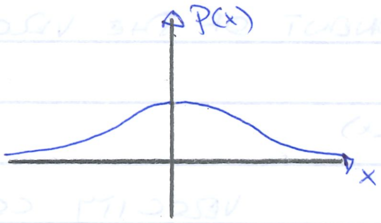
A35 NON-GAUSSIAN STATISTICS

TURBULENCE IS NOT CHARACTERIZED BY GAUSSIAN DISTRIBUTION. TURBULENT FLOWS DONT FOLLOW GAUSSIAN STATISTICS.

DISTRIBUTION $p(x)dx$ IF IT IS A GAUSSIAN ALL YOU NEED TO KNOW IS $\langle \underline{x}^2 \rangle \rightarrow p(x) = \frac{1}{\sqrt{2\pi \langle x^2 \rangle}} \cdot e^{\left(-\frac{x^2}{2\langle x^2 \rangle}\right)}$

$$\langle x^2 \rangle = \int dx \, p(x) \cdot x^2$$

IF WE MEASURE HIGHER ORDER MOMENTS IN GAUSSIAN:



$$\langle x^{2n+1} \rangle = 0$$

$$\langle x^{2n} \rangle = (2n+1)!! \langle x^2 \rangle^n$$

$$(2n)!! = n \cdot (n-2)!!$$

$$4!! = 4 \cdot 2 = 8$$

$$4! = 4 \cdot 3 \cdot 2 = 24$$

$$\langle x^n \rangle = \int dx \, p(x) \cdot x^n$$

$$\varnothing(t) = \int dx \, p(x) \cdot \exp(itx) = \int dx \, p(x) \cdot \underbrace{\sum_n \frac{(itx)^n}{n!}}_{\text{SERIES EXPANSION}} = \sum_n \frac{(it)^n}{n} \cdot \int dx \, p(x) \cdot x^n >$$

$$= \sum_n \frac{(it)^n}{n!} \langle x^n \rangle$$

CHARACTERIZE NON-GAUSSIAN STATISTICS BY HIGHER ORDER MOMENTS:

$$S_{ijk}(\vec{r}) = \langle v_i(x) \, ; \, v_j(x) \, ; \, v_k(x+r) \rangle$$

DEVIDE OUT SCALING

$$v^3 k(r) = \langle v_x^2(x) \cdot v_x(x + e_x \cdot r) \rangle$$

A36 GENERAL SLOPES OF VELOCITY TENSOR

(HOMOGENOUS & ISOTROPIC)

$$v_i = A(r) \cdot r_i$$

$$Q_{ij} = A(r) \cdot r_i r_j + B(r) \cdot \delta_{ij}$$

$$S_{ijk} = A(r) \cdot r_i r_j r_k + B(r) r_i \delta_{jk} + C(r) \cdot r_j \delta_{ik} + D(r) \cdot r_k \delta_{ij}$$

Q_{ij} = $\underline{A}$ AND $\underline{B}$ MUST BE RELATED TO f AND g

$$B = v^2 \cdot g \quad A = \frac{v^2}{r^2} \cdot (f-g) \rightarrow Q_{ij} = v^2 \cdot \left(\frac{f-g}{r^2} \cdot r_i r_j + g \delta_{ij} \right)$$

$$\text{if } \operatorname{div} \vec{v} = 0 : \frac{\partial}{\partial r_i} Q_{ij} = 0 = \underbrace{\left(r A' + 4A + \frac{B'}{r} \right)}_{=0} \cdot r_j = 0$$

SUBSTITUTE f AD g : $g = f + \frac{r}{2} f'$

$$\rightarrow Q_{ij} = \frac{v^2}{2r^2} \cdot [(r^2 f)' \cdot \delta_{ij} - f' r_i r_j$$

IN ISOTROPIC INCOMPRESSIBLE TURBULENCE $R = \operatorname{tr} Q$ DETERMINE

~~$R = \frac{v^2}{2r} \cdot (r^3 f')$ DEFINE $e = \int dr f$~~

THE CORRELATIONS COMPLETELY:

$R = \frac{v^2}{2r} (r^3 f)$ DEFINE $e = \int dr f$ (INTEGRAL SCALE OF TURBULENCE)

$$v^2 \cdot e = v^2 \int dr\, f(r) = \int_0^\infty dr\, R(r)$$

A37 DYNAMICS OF TURBULENCE → KARMAN - HOWARTH EQ.

- USES NAVIER-STOKES-EQUATION FOR PREDICTING THE TIME EVOLUTION OF Q_{ij}

FLOW AT

$$\frac{\partial}{\partial t} v_i = - \frac{\partial}{\partial x_k} (v_i v_k) - \frac{1}{\rho} \frac{\partial}{\partial x_i} p + \mu \cdot \Delta v_i \quad @ \vec{x} \quad | \cdot v_j'$$

$$\frac{\partial}{\partial t} v_j' = - \frac{\partial}{\partial x_k'} (v_j' v_k') - \frac{1}{\rho} \frac{\partial p'}{\partial x_j'} + \mu \Delta' \vec{x}' \quad @ \vec{x}' \quad | \cdot v_i$$

NOW WE ADD BOTH EQUATIONS TOGETHER

$$\frac{\partial}{\partial t}\langle v_i v_j'\rangle = -\langle v_i \frac{\partial}{\partial x_k'}(v_j' v_k') + v_j' \frac{\partial}{\partial x_k}\cdot(v_i v_k)\rangle -$$

$$-\frac{1}{\rho}\langle v_i \frac{\partial}{\partial x_i'} p' + v_j' \frac{\partial p}{\partial x_i}\rangle +$$

$$+\mu \cdot \langle v_i \Delta v_j' + v_j \Delta v_i'\rangle$$

LET'S SIMPLYFY THIS A LITTLE BIT $\frac{\partial}{\partial t}$; Δ AND $\frac{\partial}{\partial t}$ COMULATE

WITH $\langle..\rangle$ (BOTH LINEAR) $\frac{\partial}{\partial x_i}\langle ..\rangle = \frac{\partial}{\partial r_i}\langle ..\rangle$

v_i DOES NOT DEPEND ON x' AND v_j' NOT ON $\vec{x}$

$$\rightarrow \frac{\partial}{\partial t} Q_{ij} = \frac{\partial}{\partial r_k}(S_{ikj} + S_{jki}) + 2\mu \cdot \Delta Q_{ij}$$

BECAUSE $\langle v_i p'\rangle = \langle v_j' p\rangle = 0$

NOW LET'S SUBSTITUTE Q_{ij} AND S_{ijk} IN TERMS OF f AND k:

$$\frac{\partial}{\partial t}(v^2 r^4 f) = v^3 \frac{\partial}{\partial r}(r^4 k) + 2\mu v^2 \cdot \frac{\partial}{\partial r}(r^4 f')$$

THIS IS KARMAN-HOWARTH EQUATION

KARMAN-HOWARTH EQUATION DESCRIBES EVOLUTION OF VELOCITY CORRELATIONS

- GENERALISM TO THE n-TH MOMENT
- EVOLUTION OF $\langle v^n\rangle$ DEPENDS ON $\langle v^{n+1}\rangle$
- GENERATION OF SKEWNESS $\langle v^3\rangle$ FROM $\langle v^4\rangle \simeq 3\langle v^2\rangle^2$
- $(\frac{\partial}{\partial t} + \Delta)$ DIFFUSION

KARMA-HOWART EQUATION IN TERM OF R

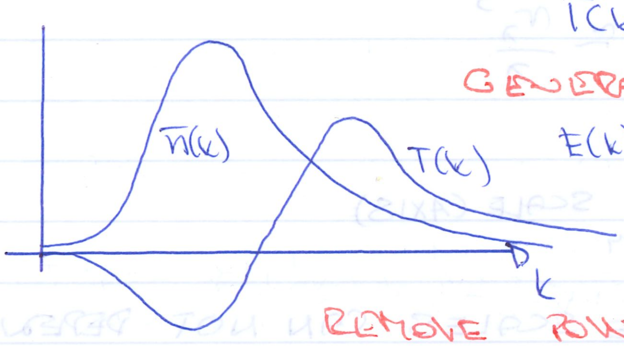
$$\frac{\partial}{\partial t} R = \Gamma(r) + 2\mu \Delta R \text{ WITH } \Gamma = \frac{1}{2r^2}\frac{\partial}{\partial r} \cdot \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r^4 u^3 k\right)}_{\text{EXPANSION}}$$

INTRODUCE ENERGY SPECTRUM: $E(k) = \frac{2}{\pi} \cdot \int dr\, R(r) \cdot (kr) \sin(kr) \rightleftharpoons R(t) = \int dk \cdot E(k) \cdot \frac{\sin(kr)}{kr}$

$$\to \frac{\partial}{\partial t} E = T(k) - 2\mu k^2 E(k)$$

T ~ SPECTRAL ENERGY TRANSFER FUNCTION

$$T(k) = -\frac{\partial}{\partial k}\Pi$$



GENERATE POWER ON SMALL SCALES

$E(k) \sim k^{-5/3}$

KOLMOGOROV-SPECTRUM

REMOVE POWER ON LARGE SCALES

A39 TURBULENCE SUMMARY

	DYNAMICS	SPECTRUM	CORRELATION
QUANTITY	REYNOLDS TENSOR $\tau_{ij} \langle u_i' u_j' \rangle$	ENERGY SPECTRUM $E(k)$	FOURIER TRANSFORM VELOCITY CORRELATION $Q_{ij} = \langle u_i(x) u_j(x+r) \rangle$
PHENOMENON	TURBULENT PREASSURE $1/3\, tr(D \cdot \epsilon)$ BOUSSINESQ APPROXIMATION (MODEL)	KOLMOGOROV CASCADING	GENERATION OF NON-GAUSSIAN STATISTICS
EVOLUTION	REYNOLDS AVERAGE NAVIER-STOKES EQUATION	SPECTRAL ENERGY FLUX Π	KARMAN HOWARTH EQUATION

A40 TURBULENCE: KOLMOGOROV CASCADING

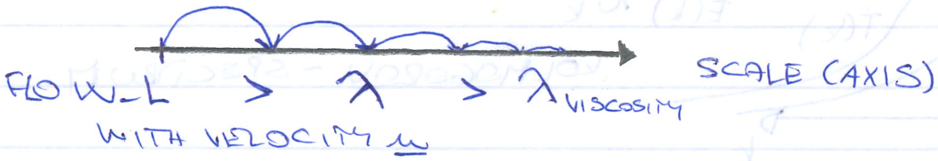
- VORTICITIES ARE UNSTABLE NOW THE KELVIN-HELMHOLTZ-INSTABILITY
  → BREAKUP INTO SECONDARY VORTICIES

- PROCESS AND A CASCADE OF VORTICITIES AT EVEN SMALLER SCALES IS ESTABLISHED
  $\longrightarrow$ KOLMOGOROV CASCADE

- ON THE SMALLEST SCALES THE REYNOLDS NUMBER BECOMES SMALL AND VISCOSITY DISSIPATES THE VORTEX

LOOK AT VORTICITIES OF SITE λ, WITH VELOCITY $v_\lambda \cong \omega\cdot\lambda$, KINETIC ENERGY $\epsilon \sim \frac{1}{2} v_\lambda^2$, & ENERGY DISIPATION:

$$\dot{E} = \frac{d}{dt}E = \frac{\Delta E}{\Delta t} = \frac{v_\lambda^2}{2}\cdot\frac{v_\lambda}{\lambda} \simeq \frac{v_\lambda^3}{\lambda}$$



ENERGY FLOW THROUGH THE SCALES CAN NOT DEPEND ON λ:

$\longrightarrow$ OTHERWISE THERE WOULD BE AN ACCUMULATION

$\longrightarrow$ ENERGY AS A FUNCTION OF SCALE SHOULD BE A POWER LAW.

$\dot{\epsilon} \simeq \frac{v_\lambda^3}{\lambda}$ IF $\dot{\epsilon}$ IS CONSTANT AND NOT A FUNCTION OF SCALE THEN $v_\lambda \sim \lambda^{1/3}$; $v(\lambda) = u\left(\frac{\lambda}{L}\right)^{1/3}$,

$$\omega = \frac{v_\lambda}{\lambda} = \frac{u}{\lambda}\cdot\left(\frac{\lambda}{L}\right)^{1/3} = u\cdot\left(\frac{1}{(\lambda^2\cdot L)^{1/3}}\right)$$

WHILE THE VELOCITY DECREASES TOWARDS λ_{visc}, THE VORTICITY INCREASES.

VISCOUS FORCES DISSIPATE ENERGY AND GENERATE HEAT

$$\eta\cdot\left(\frac{v_\lambda}{\lambda}\right)^2 \sim \eta\cdot\left(\frac{v_\lambda^3}{\lambda}\right)^{2/3}\cdot\lambda^{-\frac{4}{3}} = \eta\cdot\dot{\epsilon}^{2/3}\cdot\lambda^{-\frac{4}{3}} = \rho\cdot\dot{\epsilon}_{visc}$$

STEADY SITUATION ENERGY FLOW = DISSIPATION RATE

SET $\lambda = \lambda_{visc}$ $\quad \lambda_{visc} = \left(\frac{\eta\cdot L^{1/3}}{\rho\cdot u}\right)^{3/4} = L\cdot\left(\frac{\mu}{u\cdot L}\right)^{3/4} = \frac{L}{Re^{3/4}}$

KOLMOGOROV SPECTRUM

$$\langle v_\lambda^2 \rangle \sim \lambda^3 \cdot \lambda^{2/3} = \lambda^{11/3} = k^{-\frac{11}{3}}$$ WITH WAVENUMBER $k = \frac{2\pi}{\lambda}$

FROM FOURIER $\nearrow$ $\qquad v_\lambda \sim \lambda^{1/3}$

FOR ENERGY

$$E(k) = k^2 \cdot k^{-\frac{1}{3}} = k^{-5/3}$$

KOLMOGOROV SCALING

A41 $\overset{\text{TIME-}}{\vee}$ REVERSIBILITY OF THE EULER-EQUATION

$$t \to -t \; ; \; \vec{v} \to -\vec{v} \; ; \; \partial_t \to -\partial_t$$

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla\phi + \mu\Delta\vec{v}$$

TIME REVERSAL INVARIANT BUT A VISCOUS TERM WOULD BREAK IT.

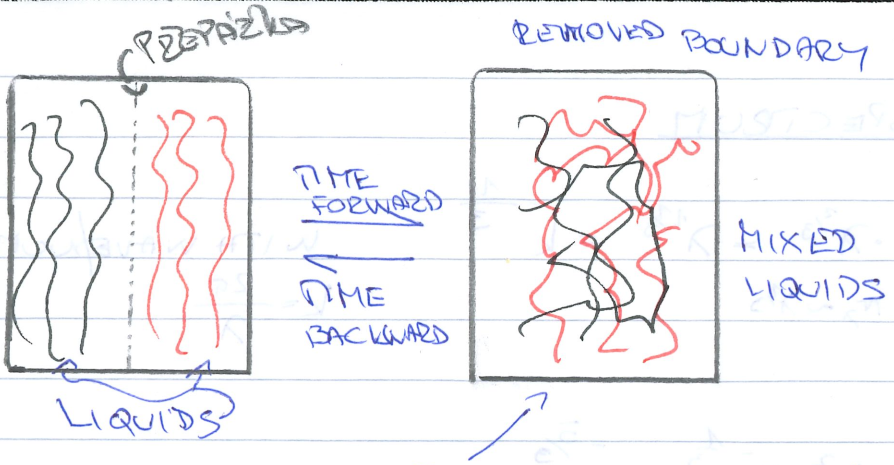
VISCOSITY → DISSIPATION OF ENERGY, HEAT CAN NOT BE CONVERTED INTO KINETIC ENERGY

VORTICITY + MOMENTUM DIFFUSION IS NOT EVOLVING BACKWARDS

WHAT HAPPENS IN TURBULENCE?

- KOLMOGOROV-CASCADING
  LARGE VORTICITIES TRANSFER POWER TO SMALL SCALES
- SPECTRAL ENERGY FLUX
- KARMAN-HOWARTH - GENERATION OF NON-GAUSSIAN FEATURES
- TURBULENT PRESSURE FROM MICROSCOPIC MOTION
- KEVIN-HELMHOLTZ-INSTABILITY

PROBABILISTIC ARGUMENT FOR MACROSCOPIC TIME-NON-REVERSIBILITY THROUGH EXP.

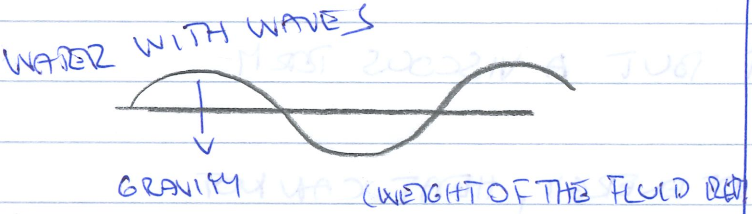


THESE LIQUIDS DON'T UNMIX, BUT THERE IS A SPECIAL INITIAL CONDITION FOR UNMIXING!

A42 WAVES & INSTABILITY

- TWO TYPES OF WAVES

GRAVITY WAVES

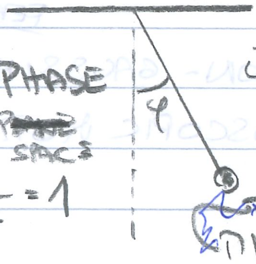


RESTORING FORCES GRAVITY & BUOYANCY (WHICH IS IN FACT GRAVITY)

RESTORE $\nabla\phi \downarrow\uparrow$ DISPLACEMENT

STABLE, OSCILLATING

PENDULUM



PHASE

ELIPSES IN ~~PHASE~~ SPACE

$$\varphi^2 + \frac{\dot{\varphi}^2}{\omega^2} = 1$$

$$\ddot{\varphi} = -\frac{g}{l}\varphi$$

$$\varphi \sim \exp(\pm i\omega t)$$

$$\omega^2 = \frac{g}{l} \rightarrow T \sim \sqrt{\frac{l}{g}}$$

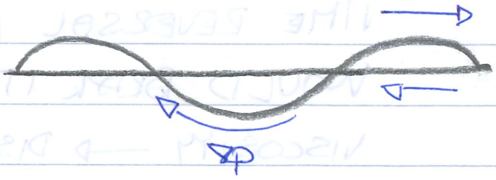
TIMESCALE

CONVEX POTENTIAL $\propto \varphi^2$

BECAUSE $\sin^2(\omega t) + \cos^2(\omega t) = 1$

INITIALLY CLOSE POINTS STAY CLOSE

KELVIN-HELMHOLTZ INSTABILITY

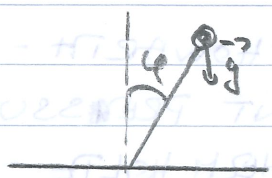


PREASSURE GRADIENT ∇p

AMPLIFIES WAVE $\nabla p \uparrow\downarrow$ DISPLACEMENT

UNSTABLE, EXPONENTIALLY GROWING.

INVERTED PENDULUM



$$\ddot{\varphi} = +\frac{g}{l}\varphi$$

$$\varphi \sim \exp(\pm \omega t)$$

$$\sinh(\omega t), \cosh(\omega t)$$

TIME SCALE IDENTICAL $\omega^2 = \frac{g}{l} \rightarrow T \sim \sqrt{\frac{l}{g}}$

HYPERBOLAE IN PHASE SPACE

$$\varphi^2 - \frac{\dot{\varphi}^2}{\omega^2} = 1$$

BECAUSE $\sinh^2(at) - \cosh(at) = 1$

SEPARATION $\propto \exp(\omega t)$ of INITIAL CLOSE POINS SEPARATES

A43 WAVES - TAXONOMY OF WAVES → DEPENDS ON RESTORING FORCE

- TYPES OF WAVES CAN BE CHARACTERIZES BY THE TYPES OF RESTORING FORCES

$$\frac{\partial}{\partial t}\vec{v}+(\vec{v}\nabla)\vec{v}=-\frac{\nabla p}{\rho}-\nabla\phi+\mu\Delta\vec{v}+2\Omega\times\vec{v}+\text{CAPILLARY}+\frac{q}{\rho}\cdot\vec{B}\times\vec{v}$$

NO

SOUND

GRAVITY

NON-NEWTONIAN ELASTIC

CORIOLIS FORCE (ROSSBY WAVES)

SURFACE TENSION

LORENZ FORCE (PLASMA)

HARMONIC WAVES → LINEAR SUPERPOSITION MUST HOLD

HARMONIC WAVES → RESTORING FORCE PROPORTIONAL TO AMPLITUDE.

VELOCITY OF THE WAVE = $\dfrac{\text{MAGNITUDE OF RESTORING FORCE}}{\text{INERTIA OF THE SYSTEM}}$

SOUND WAVES - RESTORING FORCE = PREASSURE ~~FORCE~~

PREASSURE IS LINKED TO ρ THROUGH THE EQUATION OF STATE $p=p(\rho)$.

LINEARIZATION:

$$p=p_0+\delta p \; ; \; \rho=\rho_0+\delta\rho \; ; \; \vec{v}=\delta\vec{v} \quad \text{WITH } v_0=0$$

WITH CONSTANT AND HOMOGENEOUS $p_0 ; \rho_0 ; \vec{v}_0=0$

- CONTINUITY $\frac{\partial}{\partial t}\delta\rho+\rho_0 \operatorname{div}\delta\vec{v}=0$

BECOMES. THIS $\operatorname{div}(\rho\vec{v})=\nabla\rho\cdot\vec{v}+\rho\overset{\text{div}}{\nabla\cdot}\vec{v}=0=$

$$=\underset{0}{\nabla\delta\rho\,\delta\vec{v}}+(\rho_0+\delta\rho)\operatorname{div}\delta\vec{v}\simeq\rho_0\operatorname{div}\delta\vec{v}$$

EULER EQUATION: $\frac{\partial}{\partial t}\delta\vec{r} + \frac{1}{\rho_0}\nabla\delta p = 0$ (if $\mu=0$, $\nabla\phi=0$)

BECAUSE $\frac{1}{\rho_0+\delta\rho}\cdot\nabla(p_0+\delta p) = \frac{1}{\rho_0}\nabla\delta p$

IF $\nabla p_0 = 0$ AND $\rho_0 \gg |\delta\rho|$

RECAST $\nabla\delta p$ IN TERMS OF $\nabla\delta\rho$ USE AN EQUATION OF STATE $\underbrace{\delta p = \frac{\partial p}{\partial\rho}\Big|_{\rho_0}}_{\text{DERIVATIVE, EVALUATED AT } \rho_0}\delta\rho$ AS A FUNCTION OF ρ

$\rightarrow \frac{\partial^2}{\partial t^2}\delta\rho - c^2\Delta\delta\rho = 0$ WITH $c=\sqrt{\frac{\partial p}{\partial\rho}}\Big|_{\rho_0}$ BY ELIMINATING $\frac{\partial}{\partial t}\operatorname{div}\delta\vec{r}$ FROM BOTH EQUATIONS

RESTORING FORCE IS THE ADIABATIC COMPRESIBILITY (THERE ARE TEMPERATURE CHANGES IN A SOUND WAVE, BUT THERE IS NO LOSS OF THERMAL ENERGY $\delta Q=0$

$\left(\frac{\partial p}{\partial\rho}\right)_S = \frac{c_p}{c_v}\left(\frac{\partial p}{\partial\rho}\right)_T$ WITH $\mathcal{H}=\frac{c_p}{c_v}$ ADIABATIC COEFFICIENT

@ CONST. ENTROPY ADIABATIC

@ AT CONST. TEMPERATURE ISOTERMAL

ADIABATIC SOUNDSPEED

SOUND SPEED FOR AN IDEAL GAS $c=\sqrt{\mathcal{H}\frac{RT}{\mu}}$ BECAUSE OF $p=\frac{RT}{\mu}\rho$

μ – MOLEKULAR WEIGHT

WAVE EQUATION $\frac{\partial^2}{\partial t^2}\rho - c^2\Delta\rho = 0$

c – SPEED OF SOUND $c=\sqrt{\frac{\partial p}{\partial\rho}\Big|_{\rho_0}}$

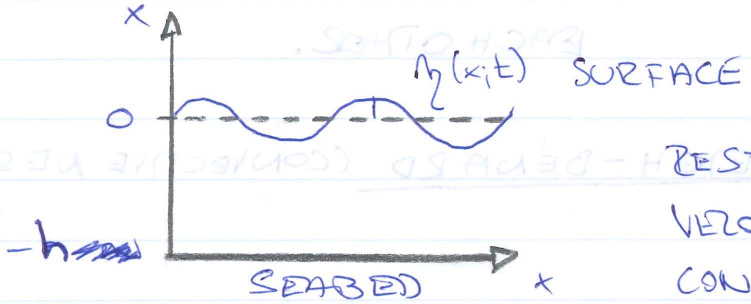
SOUND WAVE PROPAGATE ISOTROPICALY, TIME IS REVERSABLE

RELATIVISTIC WAVE – c – SPEED OF LIGHT

A44 AIRY – LAPLACE WAVES

RESTORING FORCE – GRAVITY

- THESE ARE SURFACE WAVES IN WATER – RESTORING FORCE IS SIMPLY THE WEIGHT OF THE FLUID.
- AGAIN WE USE THE VELOCITY POTENTIAL φ WITH $\vec{v} = \nabla\varphi$



RESTORING FORCE – GRAVITY (BUOYANCY)

$$\left.\begin{array}{l} \text{VELOCITY POTENTIAL } \nabla\varphi = \vec{v} \\ \text{CONSTANT DENSITY: } \operatorname{div}\vec{v} = 0 \end{array}\right\} \Delta\varphi = 0$$

$\rho = \text{KONST.}$

(INCOMPRESSIBLE FLUIDS)

BOUNDARY CONDITIONS $v_z = 0$ AT $z = -h \rightarrow \frac{\partial\varphi}{\partial z} = 0$

LET'S LOOK AT THE SURFACE (THAT'S MOVING) – THAT MOVES UP & DOWN

FREE SURFACE: $\frac{\partial}{\partial t}\eta = v_z = \frac{\partial\varphi}{\partial z}$ AT $z = \eta(x;t)$

DYNAMICS OF THE SURFACE $\frac{\partial\varphi}{\partial t} = -g\eta$ COMES FROM BERNOULI EQUATION $\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + gz = \text{KONST.}$

DYNAMIC WAVE ANZAC $\eta = \eta_0 \cdot \cos(kx - \omega t) \rightarrow$

$$\rightarrow \varphi = \frac{\omega}{k} \cdot \eta_0 \cdot \frac{\cosh[k(z+h)]}{\sinh(kh)} \cdot \sin(kx - \omega t)$$

FROM BOUNDARY CONDITION: DISPERSION $\omega^2 = gk \cdot \tanh(kh)$

~~WHERE~~ THIS IS WHAT WE GET WHEN WE SUBSTITUTE THIS TO THE WAVE EQUATION. BASED ON DEAPTH OF THE WATER h WE GET WAVES FROM DIFERENT REGIMES:

DEEP WATER: $\frac{h}{\lambda} = kh >> 1 \rightarrow \omega^2 = gk$

SHALLOW WATER $\frac{h}{\lambda} = kh << 1 \rightarrow \omega^2 = ghk^2$

THEREFORE GROUP VELOCITY $\frac{\partial\omega}{\partial k} \neq \frac{\omega}{k}$ IS UNEQUAL TO THE PHASE VELOCITY.

TSUNAMI - DEEP WATER SMALL AMPLITUDES, SHALLOW WATER LARGE AMPLITUDES.

KELVIN-HELMHOLTZ INSTABILITY - TWO LAYERS OF LIQUID AND THEY ARE SHEARED AGAINST EACH OTHER.

A45 INSTABILITIES - RAYLEIGH-BÉNARD (CONVECTIVE INSTABILITY).

HOW TERMAL CONVECTION WORK:

$$\underbrace{\frac{\partial T}{\partial t}}_{\text{TEMPERATURE CHANGE}} + (\vec{v}\nabla)T = \underbrace{\kappa\Delta T}_{\text{DIFUSION}}$$

κ - THERMAL CONDUCTIVITY

ADVECTION - DIFFUSION EQUATION FOR THE TEMPERATURE T WITH THERMAL CONDUCTIVITY κ

IN ANALOGY TO THE MOMENTUM TRANSPORT: (EULE EQ.)

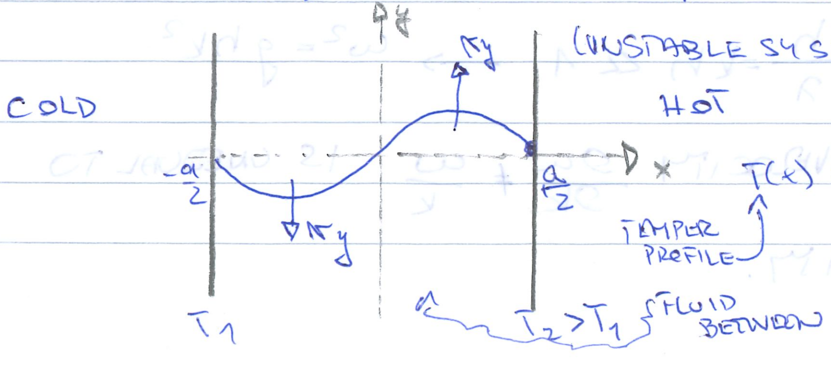
$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla\phi + \mu\Delta\vec{v}$$

TEMPERATURE VARIANCE -> CHANGES IN VOLUME:

ρ & $p(\rho;T)$ -> DEPENDS ON TEMPERATURE AND REACTS BACK ON THE FLOW

THERMAL CONVECTION: T CHANGES ρ, GIVING RISE TO THE BUOYANT FORCES

- EXAMPLE: FLUID ELEMENT BETWEEN TWO PLATES -> HORIZONTAL ∇T (UNSTABLE SYSTEM) - STABLE SOLUTION IS NOT POSSIBLE



$T(x) = T_0 + \Delta T \cdot \frac{x}{a}$;

TEMPER PROFILE

$T_0 = \frac{1}{2}(T_1 + T_2)$; $\Delta T = T_2 - T_1$

INCOMPRESSIBLE FLUIDS · ρ DEPENDS ONLY ON TEMPERATURE

$$\rho = \rho_0 + \partial\rho = \rho_0 \cdot [1 - \alpha(T - T_0)] = \rho_0 [1 - \alpha \cdot \Delta T \cdot \frac{x}{a}]$$

TEMP INCREASES FROM LEFT TO THE RIGHT, DENSITY INCREASE FROM LEFT TO RIGHT.

α - TERMAL EXPANSION COEFFICIENT

- VERTICAL LINES OF CONSTANT ρ

FLUID INITIALY AT REST

∇p_0 ~~$\cdot$~~ $= \rho \cdot \nabla\phi$ (HYDROSTATIC EQULIBRIUM

→ LINES OF CONSTANT p (PREASSURE) COINCIDE WITH EQUIPOTENTIAL LINES (HORIZONTAL); $\nabla p \times \nabla\rho$ ~~PERPENDICULAR~~ PERPENDICULAR

- GRADIENT OF PREASSURE IS IN THE SAME DIRECTION AS THE GRADIENT OF GRAV. POTENTIAL

→ ASSUME $\nabla\phi = \vec{g} = \text{KONST}$ p DEPENDS ONLY y (AXIS)
ρ DEPENDS ONLY ON x (AXIS)
HYDROSTATIC EQUILIBRIUM IS NOT POSSIBLE

TEMPRATURE GRADIENT GENERATES DENSITY GRADIENT WHICH NOT PARALEL TO THE PREASSURE GRADIENT SYSTEM WILL PICK UP SOMEKIND OF ACCELERATION AND WILL BE UNSTABLE.

- $\nabla p = \rho \cdot \nabla\phi$ ∇p DEPENDS ON x, ρ DEPENDS ON y
~ EQUATION CANNOT BE FULLFIELD

- VORTICITY GENERATION $\nabla p \times \nabla\rho \neq 0$ BECAUSE ∇p IS NOT PARALELL TO THE $\nabla\rho$ IN FACT IS PERPENDICULAR

NAVIER-STOKES EQUATION

-ASSUME STATIONARY CONDITION (ACCELERATION BALANCED BY VISCOUS FORCES)

$$-\frac{1}{\rho(x)}\cdot\frac{\partial p}{\partial y} - g + \mu\frac{\partial^2}{\partial x^2}v_y = 0 \quad \text{WITH } g = \nabla\phi = \nabla_y\phi$$

FOR THE y-COMPONENT, x+z ~ TRANSLATION INVARIANCE ASSUMPTION (PERTURBATION ANSATZ)

NOW WE WILL INTRODUCE VARIATION OF ρ : $\rho(x) = \rho_0 + \partial\rho(x)$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial y} - g + \mu\frac{\partial^2}{\partial x^2}v_y - \underbrace{\frac{\partial\rho}{\rho_0}g}_{} = 0$$

BUOYANT FORCE - APPEAR WHEN THERE IS VARIATION IN ρ

$$\rightarrow \partial\rho\cdot g = -\alpha\cdot\rho_0\cdot\partial T\cdot g$$, WITH THERMAL EXPANSION

BY SUBSTITUTING

$$\mu\frac{\partial^2}{\partial x^2}v_y + g\alpha\frac{\Delta T}{a}\cdot x = 0$$

THIS IS SOLVED BY $v_y(x) = -\frac{g\alpha}{\mu a}\cdot\frac{\Delta T}{6}\cdot x^3 + Ax + B$

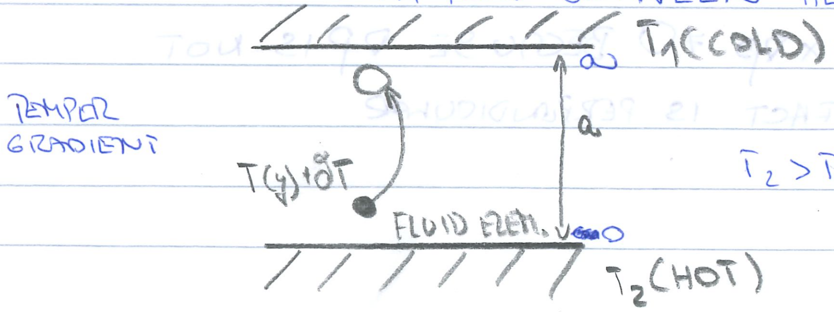
BOUNDARY CONDITION $v_y = 0$ AT $x = \pm\frac{a}{2}$

$$\rightarrow v_y(x) = -\frac{g\alpha}{\mu a}\frac{\Delta T}{6}\left(x^2 - \frac{a^2}{4}\right)$$

RAYLEIGH-BÉNARD INSTABILITY

-IS A FLOW IF IN NON-EQ?

-COMPETITION BETWEEN HEAT CONDUCTION ∝ ΔT AND ADVECTION



$T_2 > T_1$ SPHERICAL FLUID ELEMENT OF RADII r

DIFUSIVE TRANSPORT OF HEAT - TRANSPORT OF HEAT & MATTER

PHYSICAL CONDITION OF INSTABILITY - IS THE VELOCITY PERTUBERATION δv DAMPED OR WOULD IT GROW?

$$\frac{\partial}{\partial t} T + (\vec{v}\nabla) T = \mathcal{H} \Delta T$$

FLUID ELEMENT TAKES THE, IT WANTS TO INCREASE IT'S SIZE AND DECREASE ITS DENSITY IN THE PROCESS

$$\delta T = \underbrace{\frac{\partial T}{\partial y}}_{\text{TEMP GRADIENT}} \underbrace{\delta y}_{\text{DISTANCE BY WHICH THE ELEMENTS MOVE}} = \overbrace{A \cdot v \frac{r^2}{\mathcal{H}}}^{\text{TIMESCALE}} \cdot \frac{\Delta T}{a} \rightarrow \delta \rho = -\rho_0 \cdot \alpha \cdot \delta T = *$$

$$\delta y = v \cdot \tau \quad ; \quad \tau = A \cdot \frac{r^2}{\mathcal{H}}$$

TIMESCALE FOR CHANGING TEMPERATURES

$$* = -A \cdot \rho_0 \cdot \alpha \cdot v \cdot \frac{r^2}{\mathcal{H}} \cdot \frac{\Delta T}{a}$$

BUOYANT FORCE: $F_b = -\frac{4}{3}\pi \cdot r^3 \cdot \underset{\text{VARIATION IN } \rho}{\delta \rho} \cdot g = \frac{4}{3}\pi \cdot A \cdot \rho_0 \cdot \alpha \cdot \frac{r^5}{\mathcal{H}} \cdot \frac{\Delta T}{a}$

VISCOUS FORCES: $F_v = -6\pi \eta \cdot r \cdot v$

ACCELERATION IF $F_b > F_v$:

$$\frac{4\pi}{3} A \cdot \rho_0 \cdot \alpha \cdot g \cdot v \frac{r^5}{\mathcal{H}} \frac{\Delta T}{a} > 6\pi \eta \cdot r \cdot v$$

THE INSTABILITY CONDITION IS A STRONG FUNCTION OF r (BUT DOES NOT DEPEND ON v), THEREFORE LARGEST FLUID ELEMENT TO GET AN UPPER ESTIMATE OF F_b => $r = \frac{a}{2}$ (MAX POSSIBLE)

$$\underset{\text{RAYLEIGH NUMBER}}{R_a} = \frac{\alpha g \cdot a^3}{\mu \mathcal{H}} \Delta T > \underset{\text{RAYLEIGH CRITICAL NUMBER}}{R_{Ac}} = \frac{72}{A}$$

SMALL ELEMENTS EFFECTED BY SMALL BUOYANT FORCES (SMALL VOLUME)
LARGE —— " —— ACCELERATED BY BUOYANT FORCES.

FLOW BECOMES UNSTABLE IF THE VOLUME FORCE IS THE WINNING FORCE OVER THE VISCOUS FORCE.