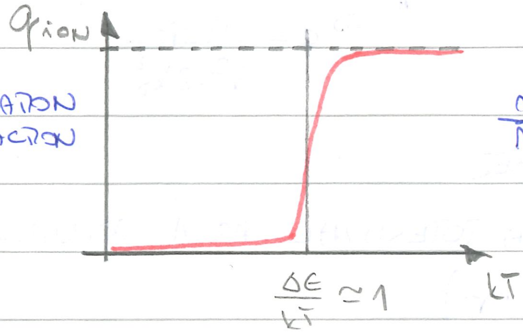
PLASMA PHYSICS

D1 PLASMA STATE

- SEPARATION OF ELECTRONS AND IONS IN A MEDIUM. RELEVANT IN ASTROPHYSICS BECAUSE TEMPERATURES ARE VERY HIGH!!!

SAHA - EQUATION: IONISATION EQUILIBRIUM OF A THERMAL PLASMA

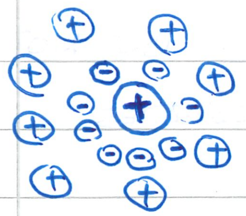
q_{ion} - IONISATION FRACTION



$$\frac{n_p \cdot n_e}{n_H} = \frac{n_p^2}{n_H} = \left(\frac{2\pi m_e}{h^2} kT\right)^{3/2} \cdot \exp\left(-\frac{\Delta E}{kT}\right)$$

DEBYE SHIELDING

- A FREE CHARGE IN A PLASMA ATTRACTS OPPOSITE CHARGES WEAKENING THE ELECTRIC FIELD COUNTERACTED BY THERMAL DIFFUSION.
- MINIMISE POTENTIAL ENERGIES $+e\phi$ AND $-e\phi$ OF THE PARTICLES IN THERMAL EQUILIBRIUM.



$$n_i = n \cdot \exp\left(+\frac{e\phi}{kT}\right)$$
$$n_e = n \cdot \exp\left(-\frac{e\phi}{kT}\right)$$

"ATMOSPHERE OF +/- CHARGE CARRIES AROUND THE MAIN CHARGE"

n - MEAN CHARGE CARRIER DENSITY

ϕ IS DETERMINED BY THE POISSON-EQUATION

$$\Delta\phi = \underbrace{4\pi e(n_i - n_e)}_{\text{CLOUD OF CHARGES (POLARISED PLASMA)}} + \underbrace{4\pi q\, \delta_D(\vec{x})}_{\text{MAIN CHARGE}}$$

WITH THE BOUNDARY CONDITION $\phi = 0$ AT INFINITY

LET'S ASSUME A HOT PLASMA ($\Delta\epsilon \ll kT$) WITH STRONG THERMAL DIFFUSION

$\Delta\epsilon \ll kT$ – BOLTZMAN-FACTOR IS CLOSE TO ONE $\Rightarrow$ TAYLOR EXPANSION

$$\Delta\phi = \underbrace{\overbrace{\frac{8\pi e^2}{kT}}^{\text{WAVE NUMBER}} n}_{\text{INVERSE LENGTH SCALE}}\phi - 4\pi q \cdot \delta_D$$

$= 2k_D$ DEBYE WAVE NUMBER

IN FOURIER SPACE

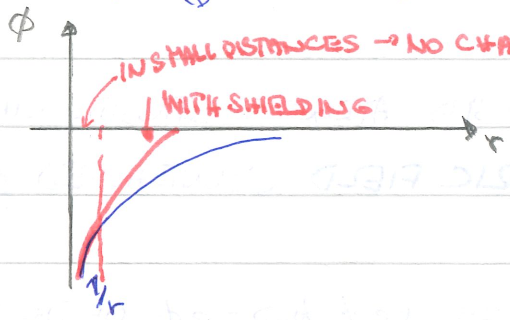
$$\left(k^2 + \frac{8\pi e^2}{kT} n\right)\phi = 4\pi q$$

$$\Rightarrow \phi = \frac{4\pi q}{k^2 + 2k_D^2}$$

WHICH HAS THE YUKAWA POTENTIAL AS A SOLUTION:

$$\underbrace{\phi(r)}_{\text{DISTANCE}} = -\underbrace{\frac{q}{r}}_{\text{COULOMB}} \cdot \exp(-\sqrt{2} \cdot k_D \cdot r)$$

IF $r \ll \lambda_D = \frac{2\pi}{k_D} \rightarrow \exp(k_D \cdot r) \simeq 1$ NO SHIELDING



AT $r \simeq \lambda_D \sim$ SHIELDING EFFECT

POTENTIAL $\rightarrow 0$ IN AN EXPONENTIAL WAVE

LENGTH SCALE $\sim$ DEBYE LENGTH

$$\frac{\lambda_D}{2\pi} = \frac{1}{k_D} = \sqrt{\frac{kT}{4\pi n \cdot e^2}}$$

λ_D DEPENDS ON $\underbrace{n, e \,\&\, T}_{\text{PLASMA PROPERTIES}}$ BUT NOT ON q

– THERMAL DIFFUSION VS. COULOMB FORCES

PLASMA FREQUENCY

– WHAT'S THE TIME NEEDED FOR CHARGE CARRIERS TO COVER λ_D IF THEY MOVE AT THERMAL VELOCITIES.

– EQUIPARTITION $v_{TH} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{kT}{m_e}} \longrightarrow t_D = \frac{\lambda_D}{v_{TH}} = \sqrt{\frac{m_e}{4\pi n e^2}}$ REACTION TIME OF THE PLASMA

$$\nu_D = \frac{1}{t_D} = \sqrt{\frac{4\pi n e^2}{m_e}} = 56\,\text{kHz}\sqrt{\frac{n}{\text{cm}^{-3}}}$$

D2 PLASMA MAGNETOHYDRODYNAMICS (MHD)

1. TREAT IONISED PLASMA AS FLUIDS
2. NO MACROSCOPIC CHARGE SEPARATION
3. NON-RELATIVISTIC MOTION

OHMS LAW $\vec{j} = \sigma \cdot \vec{E}$

ELECTRIC FIELD

CONDUCTIVITY σ

σ IS IN PLASMA USUALLY VERY LARGE $|\vec{j}| \gg |E|$ CHARGES FOLLOW $\vec{E}$ VERY FAST

$\sigma \sim \frac{1}{\text{TIME}}$

IF σ IS TO LARGE, ALL $\vec{E}$ ARE SHORT CIRCUITED, THEREFORE $\vec{E}$ ARE NOT THAT IMPORTANT IN ASTROPHYSICS.

THEN MAXWELL EQUATIONS LOOKS LIKE THIS:

$\operatorname{div} \vec{B} = 0$; $\operatorname{div} \vec{E} = 4\pi\rho$; $\operatorname{rot} \vec{B} = \frac{4\pi}{c} \vec{j}$ $\rightarrow \partial_t \vec{E} \ll \vec{j}$ BECAUSE OF THE LARGE CONDUCTIVITY

$\operatorname{rot} \vec{E} = \dots \longrightarrow$ INDUCTION EQUATIONS

OHMS LAW

$\vec{E}$ IN THE PLASMA FRAME IS $= \frac{1}{\sigma} \vec{j} \longrightarrow$ BOOST BY $\vec{\beta}$

THEN WE GET IN THE LAB. FRAME

$$\vec{E} = \frac{1}{\sigma} \vec{j} - \vec{\beta} \times \vec{B}$$

LET'S SUBSTITUT IT INTO A ACTION EQUATION:

$$\partial_{ct} \vec{B} = - \operatorname{rot} \vec{E} = \operatorname{rot}\left(\frac{1}{\sigma} \vec{j} - \vec{\beta} \times \vec{B}\right)$$

NOW WE USE AMPERS LAW: $\vec{j} = \frac{c}{4\pi} \operatorname{rot} \vec{B}$

AND THE VECTOR IDENTITY $\operatorname{rot} \operatorname{rot} \vec{B} = \nabla \underbrace{\operatorname{div} \vec{B}}_{=0} - \Delta \vec{B}$

NOW WHEN WE PUT IT TOGETHER:

$$\rightarrow \partial_{ct} \vec{B} = \frac{c^2}{4\pi\sigma} \cdot \underbrace{\Delta \vec{B}}_{\text{DIFFUSION TERM}} - \operatorname{rot}(\vec{\pi} \times \vec{B})$$ INDUCTION EQUATION

DRIVING TERM, $\propto \vec{\pi}$

DIFFUSIVITY $\frac{c^2}{4\pi\sigma} \longrightarrow D_{ct}\vec{B} = \underbrace{\frac{c^2}{4\pi\sigma}}_{\sim \frac{\text{LENGTH}^2}{\text{TIME}}} \Delta \vec{B}$

D3 EULER-EQUATION FOR A MAGNETISED PLASMA

IDEA

(1) LORENTZ-FORCES ACT ON CHARGE CARRIERS

(2) CURRENTS ARE DRIVEN BY ELECTRIC FIELDS

(3) MAGNETIC FIELDS ARE SOURCED BY CURRENTS

→ COUPLED PROCESSES IN THE EULER/CONTINUITY EQUATION

→ NEED TO INCORPORATE LORENTZ-FORCES

LORENTZ FORCE

$$F_L = \frac{e}{c}(\vec{v} \times \vec{B})$$ [IN THE GAUSS. SYSTEM OF UNITS]

LET'S MULTIPLY THIS WITH THE CHARGE CARRIER DENSITY $\vec{n}$

AND IDENTIFY $\vec{j} = \vec{n} \cdot e \cdot \vec{v}$ AS THE CURRENT DENSITY

$$\frac{e}{c} \cdot \vec{v} \times \vec{B} \cdot \vec{n} = \frac{1}{c} \cdot \vec{j} \times \vec{B} = \frac{1}{4\pi}(\mathrm{rot}\,\vec{B}) \times \vec{B}$$ SUBSTITUTE WITH AMPÈRS LAW

$$\rightarrow \rho \frac{d\vec{v}}{dt} = -\nabla p - \rho \cdot \nabla \phi + \underbrace{\frac{1}{4\pi} \cdot (\mathrm{rot}\,\vec{B}) \times \vec{B}}_{\text{LORENTZ FORCE}}$$

MATTER DENSITY

EULER EQUATION INCLUDING LORENTZ FORCE

IDENTITY $(\mathrm{rot}\,\vec{B} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{B} - \frac{1}{2}\nabla(\vec{B}^2)$

(LECTURE ABOUT VORTICITY $\vec{\omega} \times \vec{v} = (v \cdot \nabla)\vec{v} - \frac{1}{2}\nabla(\vec{v}^2)$ VORTICITY WITH $\vec{\omega} = \mathrm{rot}\,\vec{v}$)

MAGNET. PREASSURE

GRAV. POT.

$$\rightarrow \rho \frac{d\vec{v}}{dt} = -\nabla p - \rho \nabla \phi + \underbrace{\frac{1}{4\pi}(\vec{B} \cdot \nabla)\vec{B}}_{(1)} - \underbrace{\frac{1}{8\pi} \cdot \nabla(\vec{B}^2)}_{(2)}$$

TWO NEW TERMS

(1) CHARGE OF $\vec{B}$ ALONG $\vec{B}$ → PROJECTION OF $\nabla\vec{B}$ ONTO $\vec{B}$

$\sim$ PLASMA TIES TO KEEP $\vec{B}$-FIELD LINES STRAIGHT

(2) GRADIENT $\nabla(\vec{B}^2)$ OF THE FIELD ENERGY DENSITY ACTS LIKE A PREASSURE TERM, COUNTERACTING A COMPRESSION OF THE FIELD LINES → MAGNETIC PREASSURE.

D4 EQUATION OF MAGNETO-HYDRODYNAMICS

① INDUCTION EQUATION

$$\partial_t \vec{B} = \frac{c^2}{4\pi\sigma} \cdot \Delta B + \text{rot}(\vec{v} \times \vec{B})$$

② EULER-EQUATION

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla \phi - \frac{1}{4\pi\sigma}(\text{rot}\,\vec{B}) \times \vec{B}$$

③ CONTINUITY (UNCHANGED

$$\partial_t \rho + \text{div}(\rho \vec{v}) = 0$$

④ EQUATION OF STATE $p = p(\rho, T)$

IN ELECTRODYNAMICS $\text{div}\,\vec{B} = 0$ → LET'S LOOK AT $\underbrace{\text{INCOMPRESSIBLE FLUIDS}}^{\text{div}\,\vec{v} = 0}$

$$\frac{d}{dt}\vec{B} = \frac{\partial}{\partial t}\vec{B} + (\vec{v}\cdot\nabla)\vec{B} = (\vec{B}\cdot\nabla)\cdot\vec{v} + \frac{c^2}{4\pi\sigma}\cdot\Delta\vec{B}$$

$$\frac{d}{dt}\vec{v} = \frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\cdot\nabla(p + \vec{B}^2) + \frac{1}{4\pi\rho}(\vec{B}\nabla)\vec{B} + \overbrace{(\mu \cdot \Delta\vec{v})}^{\text{VISCOSITY}}$$

USING THE RELATION (REMEMBER VORTICITY LECTURES)

$$\text{rot}(\vec{A}\times\vec{B}) = \vec{A}\underbrace{\text{div}\,\vec{B}}_{=0} - \vec{B}\underbrace{\text{div}\,\vec{A}}_{=0} + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\nabla)\vec{B}$$

THESE TERMS VANISHES DUE TO INCOMPRESSIBILITY AND ABSENCE OF MONOPOLES.

2 COUPLED, NONLINEAR EQUATIONS → MHD IS DIFFICULT

D5 ADVECTION AND DIFFUSION SCALES IN MHD

IN PLASMA → σ - LARGE

FLUXFREEZING: DIFFUSION VANISHES IF THE CONDUCTIVITY σ IS LARGE

→ THE $\vec{B}$-FIELD MOVES ALONG WITH $\vec{v}$

MAGNETIC REYNOLDS NUMBER (WHICH IS USUALLY LARGE!)

$\text{rot}\, \vec{v} \times \vec{B} \sim \frac{v \cdot B}{L}$ AS. $\frac{c^2}{4\pi\sigma} \Delta B \sim \frac{c^2}{4\pi\sigma} \cdot \frac{B}{L^2}$

RATIO OF THESE EQUATIONS..

LARGE R ADVECTION DOMINATES

RATIO $R = 4\pi\sigma \frac{vL}{c^2}$

MAGNETIC REYNOLDS NUMBER

SMALL R: DIFFUSION DOMINATES

INDUCTION EQUATION $\partial_t \vec{B} = \frac{c^2}{4\pi\sigma} \Delta \vec{B}$; DIFFUSION OF $\vec{B}$-FIELDS

IF THE CONDUCTIVITY IS LARGE DIFFUSION DOES NOT PLAY A ROLE 'FLUX FREEZING'

MAG. REY. NUM. R - DESCRIBES RELATIVE EFFECTIVNESS OF ~~DEDUCTIONS~~ ADVECTION & DIFFUSION

D6 GENERATION OF MAGNETIC FIELDS → "BATTERY MECHANISM"

- HOW TO GENERATE MAGNETIC FIELDS → ELECTRIC CURRENT - SEPARATION OF CHARGE
  → TIME CHANGING ELECTRIC FIELDS

① IN ORDER A CURRENT THAT CAN INDUCE A MAGNETIC FIELD, ONE NEEDS A CHARGE SEPARATION

② BUT NEGATIVE AND POSSITIVE CHARGES IN A PLASMA ARE TIGHTLY COUPLED, DUE TO COULOMB FORCES.
→ NEED TO RELAX THIS IS AN ASSUMPTION.

③ THE INDUCTION EQUATION IS A DIFFUSION-ADVECTION EQUATION WITH NO SOURCE FIELDS: IT CAN WORK ON EXISTING MAGNETIC-FIELDS.

→ DROP THE ASSUMPTION ON TIGHT COUPLING TO MAKE POSITIVE AND NEGATIVE CHARGES AT DIFFERENT SPEEDS.
→ THE EFFECTIVE CURRENT SOURCES A MAGNETIC $\vec{B}$-FIELD

STARTING POINT – TWO EULER EQUATIONS FOR + (POSSITIVE) AND – (NEGATIVE) CHARGES

- ASSUME MACROSCOPIC CHARGE NEUTRALITY $n_+ = n = n_-$
- DENSITIES $\rho_\pm = n_\pm \cdot m_\pm = n \cdot m_\pm$ WITH PARTICLE MASSES

$$\begin{cases} \underbrace{n \cdot m_+}_{\rho_+} \cdot \frac{d\vec{v}_+}{dt} = -\overbrace{\nabla P_+}^{\text{GRADIENT PREASSURE}} + ne \cdot \left(\vec{E} + \frac{1}{c} \cdot \vec{v}_+ \times \vec{B}\right) - n \cdot m_+ \cdot \nabla\phi \quad / \cdot \frac{m_-}{n} \\ \underbrace{n \cdot m_-}_{\rho_-} \cdot \frac{d\vec{v}_-}{dt} = -\nabla P_- - ne \cdot \left(\vec{E} + \frac{1}{c} \cdot \vec{v}_- \times \vec{B}\right) - n\, m_- \cdot \nabla\phi \quad / \cdot \frac{m_+}{n} \end{cases}$$

NOW, LET'S CONSTRUCT A DIFFERENCE, SO WE ARE AFTER RELATIVE ACCELERATION:

$$\frac{d}{dt}\left(\vec{v}_- - \vec{v}_+\right) = -\frac{m_+}{n} \nabla P_- + \frac{m_-}{n} \nabla P_+ - \frac{e}{m_-}\left(\vec{E} + \frac{1}{c}\vec{v}_+ \times \vec{B}\right) + \frac{e}{m_+}\left(\vec{E} + \frac{1}{c}\vec{v}_- \times \vec{B}\right)$$

IF $m_+ \gg m_-$ (AS IN THE CASE OF IONS AND ELECTRONS

$$\frac{d}{dt}\left(\vec{v}_- - \vec{v}_+\right) = -\frac{\nabla P_-}{n\, m_-} - \frac{e}{m_-}\left(\vec{E} + \frac{1}{c}\vec{v}_- \times \vec{B}\right)$$

- BECAUSE ACCELERATION ON THE POSSITIVE CHARGE CARRIES ARE MUCH SMALLER DUE TO THEIR HIGH MASS.

LET'S INTRODUCE A COLLISION TERM SOURCING THE VELOCITY DIFFERENCE

$$\frac{d}{dt}\left(\vec{v}_- - \vec{v}_+\right) = -\frac{\nabla P_-}{n \cdot m_-} - \frac{e}{m_-} \cdot \left(\vec{E} + \frac{1}{c}\vec{v}_- \times \vec{B}\right) - \frac{\vec{v}_- - \vec{v}_+}{\underset{\text{TIMESCALE } \tau}{\tau}}$$

NET CURRENT DENSITY $\vec{j} = n \cdot e\,(\vec{v}_+ - \vec{v}_-)$

- RELATIVE ACCELERATION = 0 IN A STEADY STATE

$$\frac{d}{dt}\left(\vec{v}_- - \vec{v}_+\right) = 0 \rightarrow \vec{E} = -\frac{\nabla P_-}{ne} - \frac{\vec{v}_-}{c} \times \vec{B} + \frac{m_-}{ne^2 \tau} \cdot \vec{j}$$

MAKE THIS $\vec{E}$ FIELD INDUCE A MAGNETIC FIELD

$\partial_{ct}\vec{B} = -\mathrm{rot}\,\vec{E}$ (FARADAY'S LAW)

$$\rightarrow \partial_t \vec{B} = \frac{c}{e}\,\mathrm{rot}\left(\frac{\nabla p_-}{n_- e}\right) + \left(\vec{v}_- \times \vec{B}\right) - \frac{m_- \cdot c}{e^2 \cdot \tau} \cdot \mathrm{rot}\left(\frac{\vec{j}}{n}\right)$$

QUITE SIMILARLY TO THE DISCUSSION OF VORTICITY (TIME DERIVATIVE $\partial_t \omega$)

GENERATION OF MAGNETIC FIELDS: 2 RELEVANT TERMS:

① $\mathrm{rot}\left(\frac{\nabla p_-}{n}\right) = -\frac{\nabla p_- \times \nabla n}{n^2} \sim \nabla p_- \times \nabla n$ "BAROCLINIC TERM"

② $\mathrm{rot}\left(\frac{\vec{j}}{n}\right) = \frac{c}{4\pi n}\,\mathrm{rot}\,\mathrm{rot}\,\vec{B} + \underbrace{\vec{j} \times \frac{\nabla n}{n^2}}_{=0 \text{ BECAUSE } \vec{j} \parallel \nabla n} = \frac{c}{4\pi n} \cdot \mathrm{rot}\,\mathrm{rot}\,\vec{B}$

$$\rightarrow \partial_t \vec{B} = \frac{c^2}{4\pi\sigma}\Delta\vec{B} + \mathrm{rot}\left(\vec{v}_- \times \vec{B}\right) - \frac{c}{e n^2}\left(\nabla p_- \times \nabla n\right)$$

COMPARE TO THE VORTICITY EQUATION FROM FLUID MECHANICS

$\partial_t \omega = \nu \cdot \Delta\vec{\omega} + \mathrm{rot}\left(\vec{v} \times \vec{\omega}\right) - \frac{\nabla p \times \nabla \rho}{\rho^2}$ ~ FORMAL ANALOGY WITH VORTICITY GENERATION OF FLUID MECHAN.

- BAROCLINIC TERM SOURCES $\vec{B}$
- DIFFUSION OF THE MAGNETIC FIELD, VORTICITY ~ CONDUCTIVITY
- CORIOLIS OR CENTRIFUGAL FORCES CAN SOURCE $\vec{B}$
  - BIERMANN BATTERY
- SIMPLE EQUATIONS OF STATE $p = p(n)$ WILL NOT SOURCE $\vec{B}$

$$p(n) \rightarrow \nabla p \times \nabla n = \frac{\partial p}{\partial n}\underbrace{\nabla n \times \nabla n}_{0} = 0$$

D7 HYDROMAGNETIC WAVES

- PLASMA CAN CARRY WAVELIKE EXCITATIONS
- LINEARISATION (TO ENSURE SUPERPOSITION) → SIMPLE DIFF. STRUCTURE
- NEW RESTORING FORCES DUE TO MAGNETIC FIELDS

LET'S SET CONDUCTIVITY $\sigma \to \infty$: NO DIFFUSION, PERFECT FREEZING

① $\mathrm{div}\,\vec{B} = 0$

② $\partial_t \vec{B} = \mathrm{rot}(\vec{v} \times \vec{B})$

③ $\partial_t \rho = -\mathrm{div}(\rho \cdot \vec{v})$ [CONTINUITY EQUATION]

④ $\partial_t \vec{v} + (\vec{v}\,\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}(\mathrm{rot}\,\vec{B}) \times \vec{B}$ [EULER-EQUATION]

LET'S INTRODUCE PERTURBATION + LINEARISE:

$\rho = \rho_0 + \partial\rho$; $\vec{B} = \vec{B}_0 + \partial\vec{B}$; $p = p_0 + \partial p$; $\vec{v} = \partial\vec{v}$ [PERTURBATION BY CHOICE OF FRAME]

① $\mathrm{div}\,\partial\vec{B} = 0$

② $\partial_t \partial\vec{B} = \mathrm{rot}(\partial\vec{v} \times \vec{B}_0)$

③ $\partial_t(\partial\rho) = -\rho_0\,\mathrm{div}(\partial\vec{v})$

④ $\partial_t(\partial\vec{v}) = -\frac{1}{\rho_0}\cdot\underbrace{\nabla\partial p}_{} + \frac{1}{4\pi\rho_0}\mathrm{rot}(\partial\vec{B}) \times \vec{B}_0$

$\quad \to \partial p = c_s^2 \cdot \partial\rho$ ← WITH THE ADIABATIC SOUND SPEED

NOW LET'S SUBSTITUTE A PLANE-WAVE (ANZATZ FOR ALL QUANTITIES)

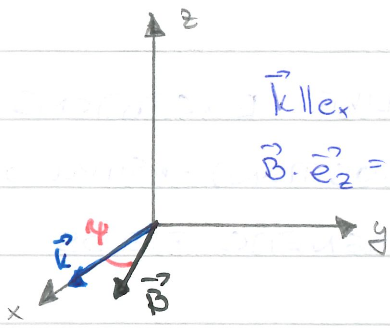
① $\vec{k} \cdot \partial\vec{B} = 0$

② $-\omega \cdot \partial\vec{B} = \vec{k} \times \partial v \times \vec{B}_0$

③ $-\omega\,\partial\rho = -\rho_0 \cdot \vec{k} \cdot \partial\vec{v}$

④ $-\omega\,\partial\vec{v} = -\frac{c_s^2}{\rho_0}\cdot\partial\rho\cdot\vec{k} - \frac{1}{4\pi\rho_0}(\vec{k} \times \partial\vec{B}) \times \vec{B}_0$

COORDINATE FRAME



$$\left.\begin{array}{l}\vec{k} \parallel e_x \\ \vec{B}\cdot\vec{e_z} = 0\end{array}\right\} \vec{k}\cdot\vec{B_0} = k\cdot B_0 \cos\psi$$

$\vec{B}$ IS IN THE xy PLANE

IN COMPONENTS WE HAVE NOW THE RELATIONS

EULER EQ. →

$$\left\{\begin{array}{l} \delta v_x - \frac{k^2}{\omega^2}\cdot c_s^2 \cdot \delta v_x = \frac{1}{4\pi\rho_0}\cdot\frac{k}{\omega_0} B_0 \delta B_y \cdot \sin\psi \\ \delta v_{yz} = -\frac{1}{4\pi\rho_0}\frac{k}{\omega}\cdot B_0 \cdot \delta B_{y,z}\cdot\cos\psi \\ \delta\rho = \rho_0\cdot\frac{k}{\omega}\cdot\delta v_x \end{array}\right.$$

HERE IS COUPLE VELOCITIES

VELOCITIES

→ $c_s^2 = \frac{\partial p}{\partial \rho}$ ADIABATIC SOUND SPEED

→ $c_k = \frac{\omega}{k}$ PHASE VELOCITY OF THE WAVES

→ $c_A^2 = \frac{B_0^2}{4\pi\rho_0}$ ALFÉN VELOCITY, FOR WAVES DRIVEN BY MAG. PREASSURE.

$$\begin{pmatrix} c_k^2 - c_s^2 - c_A^2 \sin^2\psi & c_A^2 \sin\psi\cos\psi & 0 \\ c_A^2 \sin\psi\cos\psi & c_k^2 - c_A^2\cos^2\psi & 0 \\ 0 & 0 & c_k^2 - c_A^2\cos^2\psi \end{pmatrix} \delta\vec{v} = \vec{0}$$

$\det(\ldots) = 0$ FOR NONTRIVIAL SOLUTIONS

THE POLYNOMIAL ESTABLISHES A DISPERSION RELATION BY SETTING c_k IN RELATION WITH c_s AND c_A. PROPAGATION IS DIRECTION DEPENDENT.

D8 PLASMA WAVES

$\det(\ldots) = 0$ GIVES THE DISPERSION RELATION

$$c_k^4 - c_k^2\cdot(c_A^2 + c_s^2) + c_A^2 c_s^2 \cdot \cos^2\psi = 0$$

$$\to c_k^2 = \frac{1}{2}\left[(c_A^2 + c_s^2) \pm \sqrt{(c_A^2+c_s^2)^2 - 4\cdot c_A^2 c_s^2\cdot\cos^2\psi}\right]$$

EXTREME CASES

① $\cos\psi = 0$ $\quad c_k^2 = c_A^2 + c_S^2$

② $\cos\psi = 1$ $\quad c_k^2 = c_S^2$ OR c_A^2, DEPENDING ON $\pm$ SIGN

PROPAGATION IS ANISOTROPIC

D9 ELECTRODYNAMICS OF A WAVE IN PLASMA

→ PLASMA REACTS TO THE WAVE BY BEING POLARISED →

PLASMASMA REACTS —— " —— MOTION OF THE CHARGE CARRIERS: THIS LEADS TO AN ELECTRIC FIELD $\vec{D} = \epsilon \cdot \vec{E}$ (DIELECTRIC CONST, PERMITTIVITY)

AND A MAGNETIC FIELD $\vec{B} = \mu \vec{H}$ (PERMEABILITY)

→ DIELECTRIC ε AND PERMEABILITY μ CONSTANT ARE DETERMINED BY THE MEDIUM.

→ ANISOTROPY (OF MEDIUM): ε AND μ ARE TENSORS, NOT ONLY SCALAR NUMBERS.

MAXWELL'S EQUATIONS IN MEDIUM

$\operatorname{div} \vec{D} = 0$ (SET $\rho = 0$, OTHERWISE $\operatorname{div} \vec{D} = 4\pi\rho$)

$\operatorname{div} \vec{B} = 0$

$\partial_t \vec{D} = c \cdot \operatorname{rot} H + 4\pi \vec{j}$

$\partial_t \vec{B} = -c \cdot \operatorname{rot} \vec{E}$

WITH $\vec{D} = \varepsilon \vec{E}$, $\vec{j} = \nabla \cdot \vec{E}$ (CONDUCTIVITY OF MEDIUM) AND ASSUMING $\mu = 1 \rightarrow \vec{B} = \vec{H}$

DIELECTRIC TENSOR: $\vec{D} = \varepsilon \cdot \vec{E}$; $D_i = \varepsilon_i^{\,j} \cdot E_j$

- ANZATZ $\varepsilon_{ij} = A\delta_{ij} + B \cdot \frac{k_i k_j}{k^2}$ WITH 2 FUNCTIONS $\underline{A}$ AND $\underline{B}$ (FREQUENCY OR OTHER PROP. OF THE WAVE)

(VELOCITY TENSORS → HYDRODYN. ...)

PROJECTION OPERATORS (PROJECTION IS A LINEAR OPERATION)

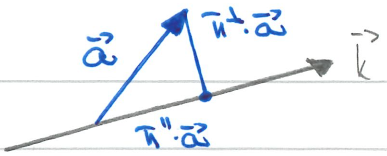
$$\pi_{ij}^{\parallel} = \frac{k_i k_j}{k^2} \; ; \quad \pi_{ij}^{\perp} = \delta_{ij} - \frac{k_i k_j}{k^2} = \delta_{ij} - \pi_{ij}^{\parallel}$$

WE HAVE A VECTOR $\underline{k}$ AND WE CAN PROJECT OTHER VECTORS ~~[illegible]~~ INTO THEIR COMPONENTS PARALELL AND PERPENDICULAR.

$n = \parallel$ ⟵ SAME!

$\pi^{\parallel} = \pi^{\parallel}$

$\pi = \pi_{\parallel}$ ⟵ SAME PI



VECTOR $\vec{k}$ AND ANOTHER VECTOR $\vec{a}$

WITH THE PROPERTIES $\pi^{\parallel}\cdot\pi^{\parallel} = \pi^{\parallel}$; $\pi^{\perp}\cdot\pi^{\perp} = \pi^{\perp}$ [IDEMPOTENCY]

$\pi^{\parallel}\cdot\pi^{\perp} = \pi^{\perp}\cdot\pi^{\parallel} = 0$ [PERPENDICULARITY]

BASIS OF ε-TENSOR $\varepsilon_{ij} = \varepsilon_{\perp}\cdot\pi^{\perp}_{ij} + \varepsilon_{\parallel}\cdot\pi^{\parallel}_{ij}$ $\longrightarrow$ LET'S SEPARATE THE AMPLITUDES!

$$tr(\varepsilon\cdot\pi^{\perp}) = tr(\varepsilon_{\perp}\cdot\pi^{\perp}\cdot\pi^{\perp} + \varepsilon_{\parallel}\cdot\overbrace{\pi^{\parallel}\cdot\pi^{\perp}}^{0}) = tr(\varepsilon_{\perp}\,\pi^{\perp}) = \varepsilon_{\perp}\,tr(\pi^{\perp}) = \varepsilon_{\perp}$$

$$tr(\varepsilon\cdot\pi^{\parallel}) = tr(\varepsilon_{\perp}\,\underbrace{\pi^{\perp}\pi^{\parallel}}_{0} + \varepsilon_{\parallel}\,\pi^{\parallel}\cdot\pi^{\parallel}) = tr(\varepsilon_{\parallel}\,\pi^{\parallel}) = \varepsilon_{\parallel}\cdot tr(\pi^{\parallel}) = \varepsilon_{\parallel}$$

TRY A WAVE $\sim \exp[i\cdot(\vec{k}\vec{x} - \omega t)]$

$$\begin{cases} \frac{\omega}{c}\vec{B} = \vec{k}\times\vec{E} \quad ; \quad \vec{k}\vec{B} = 0 \\ \frac{\omega}{c}\vec{D} = -\vec{k}\times\vec{B} \quad ; \quad \vec{k}\cdot\vec{D} = 0 \end{cases}$$ IN VACUUM

$$\longrightarrow \vec{k}\times(\vec{k}\times\vec{E}) = \frac{\omega}{c}\vec{k}\times\vec{B} = -\frac{\omega^2}{c^2}\vec{D}$$

$$\longrightarrow \vec{k}\,(\vec{k}\cdot\vec{E}) - k^2\vec{E} = -\frac{\omega^2}{c^2}\vec{D}$$

WITH DIELECTRIC TENSOR $D_i = \varepsilon_i^{\,j}\cdot E_j$

$$\longrightarrow \frac{\omega^2}{c^2}\cdot\varepsilon_i^{\,j}\cdot E_j = k^j\cdot k_j E_i - k_i k^j E_j = k^2\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)E_j$$

$$\longrightarrow \Big(\underbrace{\delta_i^{\,j} - \frac{k_i k_j}{k^2}}_{=\pi^{\perp}_{ij}} - \left(\frac{\omega}{ck}\right)^2\cdot\varepsilon_i^{\,j}\Big)\cdot E_j = 0$$

INSERT DIELECTRIC TENSOR WITH THE COMPONENTS $\varepsilon_{\parallel}$ AND $\varepsilon_{\perp}$ + CONSTRUCT DISPERSION RELATION FROM THE DETERMINANT.

$$\det\left[\left(1 - \left(\frac{\omega}{ck}\right)^2\cdot\varepsilon_{\perp}\right)\cdot\pi^{\perp}_{ij} - \left(\frac{\omega}{ck}\right)^2\cdot\varepsilon_{\parallel}\,\pi^{\parallel}_{ij}\right] = 0$$

NOW TRANSVERSE AND LONGITUDAL WAVES:

- TRANSVERSE WAVES: $\pi^{\parallel}\cdot\vec{E} = 0 \longrightarrow \det\left[\left(1 - \left(\frac{\omega}{ck}\right)^2\varepsilon_{\perp}\right)\pi^{\perp}\right] = 0$

$$\longrightarrow \omega = \frac{ck}{\sqrt{\varepsilon_{\perp}}}$$, BECAUSE $\det(\pi) = 1$

• LONGITUDAL WAVES: $\vec{n}^{\perp} \cdot \vec{E} = 0 \rightarrow \det\left[\left(\frac{\omega}{ck}\right)^2 \cdot \varepsilon_{\parallel} \cancel{\vec{n}^{\parallel}}\right] = 0$

ABY TO BYŁO NULA $\xrightarrow{\text{MUSI}}$ $\varepsilon_{\parallel} = 0$ NO LONGITUDAL WAVE PROPAGATION

D10 SHADES OF SPECTRA LINES

- WE CAN IDENTIFY CERTAIN ELEMENTS FROM THEIR SPECTRAL SIGNATURE, AND NOW WE CAN EVEN SAY SOMETHING ABOUT STATE OF MOTION OF THE SYSTEM, GRAVITATIONAL FIELDS ETC.

IDEA: RADIATION IS PRODUCED BECAUSE ATOMS ARE COUPLED TO THE EM-FIELD (RADIATION FIELD)

TRANSFER BETWEEN ATOMS/PLASMA AND THE RADIATION FIELD OR VICA VERSA → EMISSION OR ABSORPTION, DEPENDING ON THE DIRECTION OF THE ENERGY FLUX. EITHER PHOTONS → ~~ATOMS~~ MATTER OR MATTER ~~ATOMS~~ → PHOTONS

• NATURAL LINE SHAPE - IDEALISED CASE OF AN ISOLATED ATOM → TRANSITION BETWEEN TWO STATES OCCURS SPONTANEOUSLY.

$\Delta E \cdot \Delta t \sim \hbar$ $\Delta t = \Gamma^{-1}$ ~ LIFE TIME OF AN EXCITED STATE

ΔE = ENERGY UNCERTAINTY → TYPICAL WIDTH OF A SPECTRAL LINE

$E = \frac{p^2}{2m}$; $\Delta E = \frac{p}{m} \Delta p$

$p = m \cdot \frac{\Delta x}{\Delta t}$

$\Delta t = \frac{m \cdot \Delta x}{p}$

$\Delta E \Delta t = \frac{p}{m} \Delta p \cdot \frac{m}{p} \Delta x = \Delta p \Delta x \sim \hbar$

RADIATIVE TRANSITION (OSCILATION) BETWEEN TWO STATES $E_1 \rightleftarrows E_2$

$$i\hbar \partial_t a_n = \sum_m \langle n | H | m \rangle \cdot a_m \cdot \exp\left(\frac{i}{\hbar} \cdot (E_m - E_n) \cdot t\right)$$

SCHROD. EQUATION ...

FOR THE AMPLITUDE a_n OF THE STATE n. →

$$i\hbar\, \partial_t a_1(t) = \langle 1|H|2\rangle \cdot a_2(t) \cdot \exp\left(\frac{i}{\hbar}(E_2 - E_1)\cdot t\right)$$

WITH THE ANSATZ: $a_2(t) = \exp\left(-\frac{\Gamma \cdot t}{2} \cdot \exp(+i\omega t)\right)$

$$\rightarrow i\hbar\, \partial t\, a_1 = \ldots = \langle 1|H|2\rangle \cdot \exp\left[i(\omega - \underbrace{\omega_{12}}_{\rightarrow \frac{E_1 - E_2}{\hbar}})t - \frac{\Gamma}{2}t\right]$$

→ INTEGRATION FROM $\underline{0}$ TO ∞: $a_1(t=0) = 0$

$$a_1(t) = \frac{i}{\hbar}\langle 1|H|2\rangle \cdot \frac{1 - \exp\left[i(\omega - \omega_{12})\cdot t - \frac{\Gamma}{2}t\right]}{\omega - \omega_{12} + i\cdot\frac{\Gamma}{2}}$$

AT LATE TIMES $\Gamma \cdot t \gg 1$: $\exp\left(-\frac{\Gamma}{2}t\right) \rightarrow 1$

$$\rightarrow |a_1(t)|^2 = \left|\frac{\langle 1|H|2\rangle}{\hbar}\right|^2 \cdot \frac{1}{\left|\omega - \omega_{12} + i\frac{\Gamma}{2}\right|^2} = \underbrace{\left|\frac{\langle 1|H|2\rangle}{\hbar}\right|^2 \cdot \frac{1}{(\omega - \omega_{12})^2 - \frac{\Gamma^2}{4}}}_{\text{LORENTZ LINE PROFILE}}$$

PROFILE FUNCTION $\phi(\omega) = \frac{\Gamma}{(\omega - \omega_{12})^2 + \frac{\Gamma^2}{4}}$ ... WITH $\int_{-\infty}^{\infty} d\omega\, \phi(\omega) = 1$

- **COLLISIONAL BROADENING**

NATURAL LINE WIDTH ~ DETERMINED BY THE LIFE TIME OF STATES, COLLISONS CAN REDUCE THE LIFE TIME → LINE GETS BROADER

LOW PREASSURE → LOW COLLISION → LESS BROADENING

COLLISIONS CHANGE THE SCHRÖDINGER EQUATION BY A RANDOM PHASE:

$$i\hbar \cdot \partial_t a_1(t) = \langle 1|H|2\rangle \cdot \exp\left[i(\omega - \omega_{12})t - \frac{\Gamma}{2}t\right) \cdot \exp(i\phi(t))$$

$$\langle \exp(i\phi(t))\rangle = \begin{cases} \rightarrow 1 \text{ IF THERE WAS NO COLLISIONS UNTIL } \underline{t}. \\ \rightarrow 0 \text{ IF THERE WAS A COLLISION} \end{cases}$$

INTRODUCE MANY PARTICLES: EXPECTED VALUE SHOULD FOLLOW a POISSON DISTRIBUTION

$\langle \exp(i\phi(t)) \rangle \longrightarrow \exp(-\frac{\Gamma_c}{2} t) \sim$ CLASSICAL PROBABILITY

$\Delta a_1 \sim \int dt \cdot \exp[i(\omega - \omega_{12})t - \frac{\Gamma}{2} t + i\phi)$, AND AFTER AVERAGING

$\langle \Delta a_1 \rangle \sim \int dt \cdot \exp[i(\omega - \omega_{12})t - \frac{\Gamma}{2} t - \frac{\Gamma_c}{2} t]$ WITH $\exp(\frac{\Gamma_c}{2} t)$

Γ_c INCREASING THIS WITH

NO COLL. WITH COLLIS.

EFFECTIVELY $\frac{1}{\Gamma} \to \frac{1}{\Gamma} + \frac{1}{\Gamma_c} = \frac{1}{\Gamma} \cdot \frac{1}{1 + \underbrace{\frac{\Gamma_c}{\Gamma}}_{>0}}$

$\sim$ LINES GET BROADER BY COLLISIONES

- VELOCITY BROADENING

ATOMS IN MOTION $\to$ OBSERVE A DOPPLER EFFECT

$\frac{\nu}{\nu_0} \sim 1 + \beta$ WITH $\beta = \frac{v}{c}$

IF THE ATOMS HAVE VELOCITIES FOLLOWING A MAXWELL-DISTRIBUTION

$\to$ CONVOLUTION BETWEEN THE PROFILE SLOPE AND THE DISTRIBUTION

- VOIGT-PROFILE

NATURAL LINE PROFILE + COLLISIONAL BROADENING $\to$ LORENTZ

VELOCITY BROADENING $\to$ CONVOLUTION BETWEEN LORENTZ + MAXWELL

$$\phi(\omega) = \int dv \frac{\Gamma}{(\omega - \omega_{12})^2 + \frac{\Gamma^2}{4}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp(-\frac{v^2}{2\sigma^2})$$

BUT $\omega_{12} \longrightarrow \omega_{12} \cdot (1+\beta)$ WITH $\beta = \frac{v}{c}$

$$\longrightarrow \phi(\omega) = \int dV \frac{\Gamma}{\left(\omega - \omega_{12} - \omega_{12} \cdot \frac{v}{c}\right)^2 + \frac{\Gamma^2}{4}} \cdot \frac{1}{\sqrt{2\pi \sigma_v^2}} \cdot \exp\left(-\frac{v^2}{2\sigma_v^2}\right)$$

$$\phi(\omega) = \sqrt{\frac{2}{\pi}} \cdot \frac{a c}{\sigma_v \cdot \omega_{12}} \cdot \int_{-\infty}^{+\infty} dq \, \frac{\exp(-q^2)}{(u-q)^2 + a^2}$$

WITH $a = \frac{\Gamma}{\omega_{12}} \cdot \frac{c}{2\sqrt{2}\sigma}$; $q = \frac{v}{\sqrt{2}\sigma_v}$; $u = \frac{\omega - \omega_{12}}{\omega_{12}} \cdot \frac{c}{\sqrt{2}\sigma_v}$

SMALL $\Delta\omega$: $\frac{1}{(u-q)^2 + a^2} \sim 1$ $\longrightarrow$ GAUSSIAN SLOPE

LARGE $\Delta\omega$: $\exp(-q^2) \sim 1$ $\longrightarrow$ LORENTZIAN SLOPE