Advanced Quantum Field Theory

problem sheet

Lectures:

1. The Path Integral for free and interacting fields, perturbation theory, diagram expansion, correlation functions, scattering amplitudes

1. This problem is about calculating multi dimensional Gaussian integrals

$$Z_0 = \int d^N x e^{-\frac{1}{2}x^T A x}$$

where x is a $N \times 1$ column vector and A is an $N \times N$ symmetrical matrix. This can be done by finding the orthogonal matrix O that diagonalizes $A = O^T D O$ and then changing integration variables to $y = Ox$ (what happens to the Jacobian?). Show that $Z_0 = \sqrt{\frac{(2\pi)^N}{\det A}}$. Define

$$< x_{k_1} \dots x_{k_m} >= \frac{1}{Z_0} \int d^N x \; x_{k_1} \dots x_{k_m} \; e^{-\frac{1}{2}x^T A x}$$

This can be done by introducing an extra variable (column vector) j such that

$$Z_0[j] = \int d^N x e^{-\frac{1}{2}x^T A x + j^T x}$$

since then

$$< x_{k_1} \dots x_{k_m} >= \frac{1}{Z_0} \frac{d}{dj_{k_1}} \dots \frac{d}{dj_{k_m}} Z_0[j]|_{j=0}$$

where j is put to zero *after* differentiation. Find an explicit expression for $< x_{k_1} x_{k_2} x_{k_3} x_{k_4} >$ in terms of elements of the matrix A^{-1} and compare this formula with the correlation functions of a free scalar field.

2. Modify the integral in the previous problem to include "interactions"

$$Z[j] = \int d^N x e^{-\frac{1}{2}x^T A x - \frac{g}{4!}x^4 + j^T x}$$

where $x^4 = \sum_k x_k^4$. Calculate $Z[j]$ and subsequently $< x_{k_1} x_{k_2} x_{k_3} x_{k_4} >$ to first order in g.

3. For a free real scalar field in four dimension find the real and imaginary parts of the propagator. Can you reconstruct the propagator from the knowledge of the imaginary part only.

4. Solve the equation

$$\left(-\partial^2 + m^2 + \frac{\lambda}{6}\varphi^2\right)\varphi = J$$

order by order about $\lambda = 0$. Set $\varphi = \varphi^{(0)} + \lambda\varphi^{(1)} + \lambda^2\varphi^{(2)} + \ldots$. Derive explicit expressions for $\varphi^{(2)}$ and $\varphi^{(3)}$.

5. Show that

$$\Delta(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2 + m^2 - i\epsilon}$$

is a solution to

$$(-\partial_x^2 + m^2)\Delta(x - x') = \delta^4(x - x')$$

i.e. a Green function for the Klein-Gordon operator.

6. Verify that $\Delta(x - x')$ of the previous problem decays exponentially for spacelike separation.

7. Find the propagator $\Delta(x - x')$ for a $(1 + 1)$-dimensional spacetime and study the large x^1 behavior for $x^0 = 0$.

8. For the real, free Klein-Gordon field verify that

$$\langle 0 | T\varphi(x_1)\varphi(x_2) | 0 \rangle = \frac{1}{i}\Delta(x_1 - x_2)$$

and

$$\begin{aligned}\langle 0 | T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) | 0 \rangle = & - [\Delta(x_1 - x_2)\Delta(x_3 - x_4) \\ & + \Delta(x_1 - x_3)\Delta(x_2 - x_4) \\ & + \Delta(x_1 - x_4)\Delta(x_2 - x_3)]\end{aligned}$$

9. For the free, real scalar theory, let $Z_0(J) = e^{iW_0(J)}$. Evaluate the real and imaginary parts of W_0.

10. A real scalar field has a self interaction Lagrangian density

$$\mathcal{L} = \frac{1}{2} g\varphi\partial^\mu\varphi\partial_\mu\varphi \ .$$

Draw the vertex and find the associated vertex factor.

11. A complex scalar field ϕ interacts with a real scalar field φ through the interaction Lagrangian density $\mathcal{L} = g\varphi\phi^\dagger\phi$. Use a solid line for the ϕ propagator and a dashed line for the φ propagator. Draw the vertex and find the associated vertex factor.

12. A complex scalar field ϕ interacts with a real scalar field φ through the interaction Lagrangian density $\mathcal{L} = g\varphi\phi^\dagger\phi$. Assuming that $m_\varphi > 2m_\phi$, compute the total decay rate of the φ particle at tree level.

13. Consider a theory of three real scalar fields A, B and C with the interaction term $gABC$. Write down the tree-level scattering amplitutde for each of the following processes

$$\begin{aligned} AA &\to AA \\ AA &\to AB \\ AA &\to BB \\ AA &\to BC \\ AB &\to AB \\ AB &\to AC \end{aligned}$$

Your answer should take the form

$$g^2\left[\frac{c_s}{m_s^2 - s} + \frac{c_t}{m_t^2 - t} + \frac{c_u}{m_u^2 - u}\right]$$

14. Any physical consequences of a field theory should be invariant under local field redefinitions. In the real scalar theory, make a field redefinition

$$\varphi \to \varphi + \lambda\varphi^2 \ .$$

Work out the Feynman rules for the modified theory and show the tree level scattering amplitude $\varphi\varphi \to \varphi\varphi$ is zero, consistent with the fact that we are really dealing with a free theory.

2. Loop corrections to the propagator and vertices, the effective action, renormalization

1. Prove the formula

$$\frac{1}{A_1^{\alpha_1}\dots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i \alpha_i)}{\Pi_i\Gamma(\alpha_i)}\int_0^1 dx_1\dots dx_n\delta(\sum_i x_i - 1)\frac{\Pi_i x_i^{\alpha_i-1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}}$$

2. Show that $\int d^dk\, k^\mu f(k^2) = 0$ and evaluate the constant C in $\int d^dk\, k^\mu k^\nu f(k^2) = Cg_{\mu\nu}\int d^dk\, k^2 f(k^2)$. Finally evaluate $\int d^dk\, k^\mu k^\nu k^\rho k^\sigma f(k^2)$.

3. Prove

$$\int \frac{d^dk}{(2\pi)^d}\frac{(k^2)^a}{(k^2+D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{\frac{d}{2}}\Gamma(b)\Gamma(\frac{1}{2}d)}D^{-(b-a-\frac{d}{2})}$$

4. For a real scalar field with interaction $\lambda\varphi^4/4!$, draw all the contributions to the two point function $G^{(2)}$ that are of order λ^3.

5. For a real scalar field with interaction $\lambda\varphi^4/4!$, draw all the contributions to the four point function $G^{(4)}$ that are of order λ^3.

6. Show that for $a > 0$

$$\int_0^1 dx \ln\left(1+\frac{4}{a}x(1-x)\right) = -2+\sqrt{1+a}\ln\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}$$

   The let $z = \frac{4}{a}$ and study the singularity structure in the complex z-plane. Specifically find for what z the integral is real.

3. Explicit calculations in scalar field theory, dimensional regularization

1. Consider a four dimensional real scalar field with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4$$

Compute the $\mathcal{O}(\lambda)$ corrections to the propagator and compute the $\mathcal{O}(\lambda)$ terms in the coefficients of the k^2 term and the m^2 term in the expression for the self-energy $\Pi(k^2)$.

2. Repeat the calculation of the previous problem for the theory of a four dimensional complex scalar field

$$\mathcal{L} = -\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2$$

3. Calculate the $\mathcal{O}(\lambda^2)$ correction to the four point vertex for the real φ^4 theory. Choose your renormalization condition such that the coupling constant is λ when all four momenta are on shell and $s = 4m^2$. What is the $\mathcal{O}(\lambda)$ contribution to the Z factor?

4. Repeat the previous problem for the four dimensional complex ϕ^4 theory.

5. Compute for the six dimensional scalar φ^3 theory the $\mathcal{O}(\alpha)$ corrections to the two-particle scattering amplitude at threshold. That is, for $s = 4m^2$ and $t = u = 0$, corresponding to zero three-momentum for both the incoming and outgoing particles.

4. The renormalization group, effective field theory

1. Given $\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu}$, desceribe the behavior of the hypothetical field theories for which

$$\beta(\lambda_F) = \beta'(\lambda_F) = 0$$

or

$$\beta(\lambda_k) = 0 \quad \lambda_k = \lambda_F + \frac{a}{k}, \quad k = 1, 2, \ldots\infty$$

2. Show that

$$\Gamma[\varphi] = W[J] - \int d^dx\; J\varphi$$

where $J(x)$ is the solution of

$$\frac{\delta}{\delta J(x)} W[J] = \varphi(x)$$

3. Suppose that we have a set of fields $\varphi_a(x)$, and that both the classical action $S[\varphi]$ and the integration measure $\mathcal{D}\varphi$ are invariant under

$$\varphi_a(x) \to \int d^d y \; R_{ab}(x,y)\varphi_b(y)$$

for some particular function $R_{ab}(x,y)$. Show that both $W[J]$ as well as the quantum action $\Gamma[\varphi]$ are also invariant.

4. Consider performing the path integral in the presence of a background field $\bar{\varphi}(x)$. We define

$$e^{iW[J,\bar{\varphi}]} = \int \mathcal{D}\varphi \; e^{iS[\varphi+\bar{\varphi}]+i\int d^d x J\varphi}$$

Clearly $W[J,0]$ is the original $W[J]$. We also define the quantum action in the presence of the background field

$$\Gamma[\varphi,\bar{\varphi}] = W[J,\bar{\varphi}] - \int d^d x \; J\varphi$$

where now $J(x)$ is the solution of

$$\frac{\delta}{\delta J(x)} W[J,\bar{\varphi}] = \varphi(x)$$

Show that $\Gamma[\varphi,0]$ is the original quantum action and that

$$\Gamma[\varphi,\bar{\varphi}] = \Gamma[\varphi+\bar{\varphi},0]$$

5. Consider φ^4 theory

$$\mathcal{L} = -\frac{1}{2} Z_\varphi \partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2} Z_m m^2\varphi^2 - \frac{1}{4!} Z_\lambda \lambda \mu^\epsilon \varphi^4$$

in $4-\epsilon$ dimensions. Compute the beta function to $\mathcal{O}(\lambda^2)$, the anomalous dimension of m to $\mathcal{O}(\lambda)$ and the anomalous dimension of φ to $\mathcal{O}(\lambda)$.

6. Repeat the the previous problem for the complex ϕ^4 theory

$$\mathcal{L} = -Z_\phi \partial^\mu \phi^\dagger \partial_\mu \phi - Z_m m^2 \phi^\dagger \phi - \frac{1}{4} Z_\lambda \lambda \mu^\epsilon (\phi^\dagger \phi)^2$$

7. Consider the six dimensional Lagrangian density

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} Z_m m^2 \varphi^2 + Y\varphi \\ &-\frac{1}{2} Z_\chi \partial^\mu \chi \partial_\mu \chi - \frac{1}{2} Z_M M^2 \chi^2 \\ &+\frac{1}{6} Z_g \mu^{\frac{\epsilon}{2}} \varphi^3 + \frac{1}{2} Z_h h \mu^{\frac{\epsilon}{2}} \varphi \chi^2\end{aligned}$$

in $6-\epsilon$ dimensions, where φ and χ are real scalar fields, and Y is adjusted to make $\langle 0|\, \varphi(x)\, |0\rangle = 0$. Compute the one-loop contributions to each of the Z's in the $\overline{MS}$ renormalization scheme.
The bare couplings are related to the renormalized ones via

$$\begin{aligned}g_0 &= Z_\varphi^{-\frac{3}{2}} Z_g \; g \; \mu^{\frac{\epsilon}{2}} \\ h_0 &= Z_\varphi^{-1} Z_\chi^{-\frac{1}{2}} Z_h \; h \; \mu^{\frac{\epsilon}{2}}\end{aligned}$$

Define

$$\begin{aligned}G(g,h,\epsilon) &= \sum_{n=1}^{\infty} G_n(g,h)\epsilon^{-n} \equiv \ln(Z_\varphi^{-\frac{3}{2}} Z_g) \\ H(g,h,\epsilon) &= \sum_{n=1}^{\infty} H_n(g,h)\epsilon^{-n} \equiv \ln(Z_\varphi^{-1} Z_\chi^{-\frac{1}{2}} Z_h)\end{aligned}$$

By requiring g_0 and h_0 to be independent of μ, and by assuming that $\frac{dg}{d\mu}$ and $\frac{dh}{d\mu}$ are finite as $\epsilon \to 0$, show that

$$\begin{aligned}\mu \frac{dg}{d\mu} &= -\frac{1}{2}\epsilon g + \frac{1}{2} g \left(g \frac{\partial G_1}{\partial g} + h \frac{\partial G_1}{\partial h} \right) \\ \mu \frac{dh}{d\mu} &= -\frac{1}{2}\epsilon h + \frac{1}{2} h \left(g \frac{\partial H_1}{\partial g} + h \frac{\partial H_1}{\partial h} \right)\end{aligned}$$

Compute the beta functions for g and h. There will be terms of order g^3, gh^2 and h^3 in β_g and terms of order $g^2 h$, gh^2 and h^3 in β_h.

Without loss of generality we can choose g to be positive, h can then be positive or negative, and the difference *is* physically significant. For what numerical range of h/g are β_g and β_h/h both negative? Why is this an interesting question?

8. Consider a theory with a single dimensionless coupling g whose beta function takes the form

$$\beta(g) = b_1 g^2 + b_2 g^3 + \dots$$

Then consider a new definition of the coupling $\tilde{g}$ that agrees with the original definition at lowest order, so that we have $\tilde{g} = g + c_2 g^2 + \dots$. Show that $\beta(\tilde{g}) = b_1 \tilde{g}^2 + b_2 \tilde{g}^3 + \dots$.

5. Spinors, Grassmann variables, fermionic path integrals

1. Work out the Dirac equation and gamma matrices in $1+1$ and $2+1$ dimensions.

6. Explicit calculations with fermions

1. Yukawa theory is defined as a theory of a Dirac fermion and real scalar field defined by the Lagrangian

$$\begin{aligned}\mathcal{L} = & \, i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}M^2\varphi^2 \\ & + ig\varphi\bar{\psi}\gamma_5\psi - \frac{1}{4!}\lambda\varphi^4\end{aligned}$$

Derive the fermion-loop correction to the scalar propagator. Show that there is an extra minus sign for a fermion loop as compared to a scalar loop.

2. Consider changing the interaction term to $\mathcal{L}_{int} = g\varphi\bar{\psi}\psi$. Show that renormalizability require us to add a linear and and a cubic term to cancel tadpoles. Find the one-loop contributions to the renormalizing Z factor for this theory in the $\overline{MS}$ scheme.

3. For the theory in the previous problem, compute the one-loop contributions to the beta function for g, λ and κ, where κ is the coefficient of the φ^3 interaction that we had to add for renormalizability. Compute also the contributions to the anomalous dimensions of m, M, ψ and φ.

4. Consider a massless fermion field ψ coupled to a real scalar field φ by $e\varphi\bar{\psi}\psi$ in $(1+1)$-dimensional spacetime. Show that there is an effective potential generated

$$V_F = \frac{1}{2\pi}(e\varphi)^2 \ln \frac{\varphi^2}{M^2}$$

after a suitable counterterm has been added.

7. Nonabelian gauge theory

1. In ordinary Quantum Electrodynamics, show that adding a gauge fixing term

$$-\frac{1}{2}\xi^{-1}(\partial^\mu A_\mu)^2$$

to the Lagrangian results in a propagator

$$\Delta_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi\frac{k_\mu k\nu}{k^2}\right)$$

What choice of ξ corresponds to Lorenz gauge $\partial_\mu A^\mu = 0$?

2. Consider the gauge condition

$$A_i A^i = m^2$$

Discuss its validity as a gauge condition and write the corresponding path integral for Maxwell theory in this gauge.

3. Repeat the previous problem for the gauge condition

$$\partial_i A_3 \partial^i A^3 = 0$$

4. In spinor electrodynamics defined by the Lagrangian

$$\mathcal{L} = iZ_2\bar{\psi}\partial\!\!\!/\psi - Z_m\bar{\psi}\psi - \frac{1}{4}Z_3 F^{\mu\nu}F_{\mu\nu} + Z_1 e\bar{\psi}A\!\!\!/\psi$$

calculate the Z coefficients to order $\frac{e^2}{\epsilon}$ in R_ξ guage. In particular, show that $Z_1 = Z_2$ in Lorenz gauge.

5. Show that a diagram with four external photons is divergent. Why does the sum of all such diagrams have to be divergence free? (Hint: gauge invariance).

6. In non-abelian gauge theory the gauge field $A_\mu^a T^a$ transforms as

$$A_\mu(x) - \to U(x) A_\mu(x) U^\dagger(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x)$$

where $U(x) = e^{-ig\Gamma^a T^a}$. Find an expression for the infinitesimal transformations of A_μ, ϕ, $D_\mu \phi$ and $F_{\mu\nu}$.

7. Thinking of the nonabelian field strength $F_{\mu\nu}$ as a two-form, show that $\mathrm{tr}(F \wedge F)$ is closed and can be written $\mathrm{tr}(F \wedge F) = d\, \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ locally.

8. Explicit calculations in nonabelian gauge theory and the background field method

1. Derive expressions for the gauge field propagator in the guage $n^\mu A_\mu$, where n^μ is a constant vector of length one.

2. Comput the beta function for the gauge coupling in Yang-Mills theory with several Dirac fermions in the representation R_i, and several complex scalars in the representation R'_j.

3. Compute the one-loop contributions to the anomalous dimensions of m, ψ and A_μ.

4. Compute the tree-level vertex factors in background field gauge for all vertices that connect one or more external gluons with two or more internal lines (ghost or gluon).

9. Global and local symmetries, gauge invariance, gauge fixing, ghosts, BRST

1. Show that the Faddeev-Popov determinant

$$\Delta[A] = \det\left(\frac{\delta G^a}{\delta \theta^b}\right)$$

where G^a is the gauge fixing function, is gauge invariant.

2. Derive Ward identities for scalar electrodynamics in the Feynman gauge.

3. Derive Ward identities for QED in the axial gauge.

10. Anomalies

1. Consider a theory with a nonabelian gauge symmetry, and also a $U(1)$ gauge symmetry. The theory contains left-handed Weyl fields in the representations (R_i, Q_i), where R_i is the representation of the non-abelian group, and Q_i is the $U(1)$ charge. Find the conditions for this theory to be anomaly free.

2. Define the fermionic path integral measure $D\psi$ carefully by going to Euclidean space. Calculate the Jacobian upon a chiral transformation and derive the anomaly.

11. Non perturbative effects

1.

12. Exam problems

1. In four dimension, calculate the contribution to the Yang-Mills kinetic term from a non self interacting complex scalar field coupled to a non-abelian external gauge field.