The Standard Model of Particle Physics Exam problems

These are hand in assignments for the course in "The Standard Model of Particle Physics" given at Masaryk University at the fall semester 2020. They consist the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**.

1. Instead of taking the Higgs field in the two dimensional representation of $SU(2)_L$, we will choose it to be in the four dimensional representation. We start with four complex scalar fields combined into a 4×1 matrix

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} .$$

The Lagrangian density is

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi + m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 ,$$

with $m^2 > 0$ and $\lambda > 0$. This Lagrange density is invariant under the $SU(2)_L \times U(1)_Y$ transformations

$$\Phi \to e^{ig\alpha^A T_A} e^{ig' q_H \beta} \Phi ,$$

with

$$\begin{aligned} T_1 &= \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \\ T_2 &= \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix} \end{aligned}$$

$$T_3 = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

a. Verify that the generators thus defined satisfy the $SU(2)$ algebra $[T_A, T_B] = i\epsilon_{ABC}T_C$.

b. Promote this to a gauge symmetry, *i.e.* require that the theory remains invariant under the transformations but with α^A and β arbitrary function of space and time. In order to achieve this you need to introduce the gauge fields $W_\mu = W_\mu^A T_A$ and the $U(1)$ gauge field B_μ.

c. Discuss the structure of the ground state of the scalar system.

d. Show that by choosing the ground state as

$$\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix},$$

with $v = \sqrt{\frac{m^2}{\lambda}}$ we can make a gauge choice such that the scalar field is given by

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ 0 \\ \frac{1}{\sqrt{2}}(v + \sigma(x)) \end{pmatrix},$$

with $\sigma(x)$ a real scalar field and ϕ_1 and ϕ_2 two complex scalar fields.

e. Determine q_H by requiring that the photon remains massless.

f. Determine the mass ratio of the W and Z vector bosons and compare with the Standard Model result.

2. To produce Higgs bosons at the LEP accelerator one would have to have used the process $e^+e^- \to ZH$. Using tree level calculations, find the probability to produce a Higgs boson in LEP when it was running at 200 GeV.

3. Consider the decay $\pi^- \to l^-\ \bar{\nu}_l$ where $l = e$ or μ. It proceeds by the pion decaying into a W boson which weakly decays into the lepton pair. Assume that the vertex for the pion decaying into a W boson is given by

$$\frac{g}{2\sqrt{2}} f_\pi k^\mu .$$

   a) Show that (up to an irrelevant phase factor) the Feynman amplitude for this process is given by

$$\mathcal{M} = \frac{g^2}{8m_W^2} f_\pi m_\pi \bar{u}_r(p)\gamma^0(1-\gamma_5)v_s(q) \ ,$$

   where we have assumed that the lepton has momentum p and helicity r and the lepton-neutrino has momentum q and helicity s and we have neglected terms of order $\frac{m_\pi^2}{m_W^2}$ and $\frac{m_\nu}{m_l}$.

   b) Show that the decay rate for this process (summing over all final helicities and possible final momenta) is given by

$$\Gamma = \frac{g^4 f_\pi^2}{256\pi} \frac{m_l^2 m_\pi}{m_W^4} \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2 .$$

   c) Using this, calculate the ratio between the branching ratios for the decay into $e^-\bar{\nu}_e$ and $\mu^-\bar{\nu}_\mu$ where you may use

$$m_\pi = 140MeV \ , \quad m_e = 511keV \ , \quad m_\mu = 106MeV \ .$$

   and compare to the experimental result.

4. The top quark almost exclusively decays into a bottom quark and a W.

   a) Show that its decay rate is given by

$$\Gamma(t \to bW) = m_t^3 \frac{G_F}{8\sqrt{2}\pi} \left(1 - \frac{m_W^2}{m_t^2}\right)^2 \left(1 + 2\frac{m_W^2}{m_t^2}\right) ,$$

   where $G_F = \frac{\sqrt{2}g^2}{8m_W^2}$ and terms of order $\frac{m_b^2}{m_t^2}$ and $\frac{m_b^2}{m_W^2}$ have been neglected.

b) Use this result to calculate the lifetime of the top in seconds.

5. In an $SU(5)$ gauge theory let Φ^A be a scalar field in the adjoint representation of $SU(5)$. We can make it into a 5×5 matrix by contracting it with a generator in the fundamental representation of the group $\Phi = \Phi^A T_A$. In the supersymmetric scenario the potential for Φ is a cubic polynomial

$$V = a\mathrm{Tr}(\Phi^2) + b\mathrm{Tr}(\Phi^3) \ .$$

Any Φ can be diagonalized by a gauge transformation and since the potential is gauge invariant, it is clear that any extremum of the potential can be chosen to be a real, traceless diagonal matrix.

a) Find all the extrema of the potential and the unbroken gauge group associated with it.

b) Without supersymmetry one has to consider a different potential

$$V = -m^2\mathrm{Tr}(\Phi^2) + a\mathrm{Tr}(\Phi^4) + b\left[\mathrm{Tr}(\Phi^2)\right]^2 \ .$$

When is $\Phi = v \ \mathrm{diag}(2, 2, 2 - 3, -3)$ an extremum?

c) In this case, what is the masses of the fluctuations.

6. Consider a two neutrino flavor system. The mass basis states are $|\nu_1\rangle$ and $|\nu_2\rangle$ while the flavor basis states are $|\nu_e\rangle$ and $|\nu_\mu\rangle$ and the relation between them is given by a mixing angle as

$$\begin{aligned} |\nu_1\rangle &= \cos(\theta)|\nu_e\rangle + \sin(\theta)|\nu_\mu\rangle \\ |\nu_2\rangle &= -\sin(\theta)|\nu_e\rangle + \cos(\theta)|\nu_\mu\rangle. \end{aligned}$$

The mass basis states are eigenstates of the free Hamiltonian and evolve in time as

$$\begin{aligned} |\nu_1\rangle &\to e^{-iE_1 t}|\nu_1\rangle \\ |\nu_2\rangle &\to e^{-iE_2 t}|\nu_2\rangle. \end{aligned}$$

Find the probablility of finding an originally electron neutrino as still an electron neutrino after time T.

7. If QCD was based on the gauge group $SU(5)$ instead of $SU(3)$. Redo the anomaly analyzis for this case. If we want to have a "proton" in our theory, it would have to be a bound state of 5 quarks. What anomaly free choice of hyper-charges would make this true for a proton with charge $+1$?

8. Calculate and compare branching ratios for the Z boson. That is, calculate the probability that the Z boson decays to a) a pair of quarks b) an electron/positron pair and c) an electron neutrino/antineutrino pair. To simplify the calculation, assume that all final particles are massless.