Alternative mathematics?
or
Petr Vopěnka's wandering
to an alternativity and back

Alternativní matematika?
aneb
putování Petra Vopěnky k alternativitě a zpět

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Mathematics is supposed to be the most exact discipline, the queen of science. Does it mean that there is no room for alternative mathematical results, alternative truths? And if alternativity were acceptable, does this mean that mathematics also suffers from post-modern relativism?

Two stories – „case studies"

1. alternative arithmetic

Conflict between a maths teacher and pupil who insists on the "alternative fact" that 2 + 2 = 22. The teacher tries to explain the mistake, but without success. Later on, the boy's parents and the school management recognise that **everyone has the right to his or her own opinion**; therefore, it is not possible to deny that right and to insist on obsolete textbook "truths". Eventually, the teacher loses her job as she was not capable of accepting another point of view.

https://www.youtube.com/watch?v=RcJ1_eXhT1A

2 alternative higher maths?

Infinity

… is it a number, is it possible to incorporate it to math?

No

Only potential infinity: Pythagoreans, Aristotle… However, potential-DYNAMICOI infinity is not number, but a process.

Yes

Theo-Platonism

Augustine: Plato`s realm of forms-ideas to the mind of God

God knows all numbers, so he must know infinity, therefore infinity exists.

Proof or an axiom?

God knows, but mathematicians failed to implement it to maths. „Subconscious philosophy" (paradigm) did not allow them to forego self-evident axioms valid for finite world.

Axioms:

Natural, **evident** – finite world (originally on an unconscious level)

Supernatural, **weird** - infinite realm

– hidden image of omniscient being – (Theo)-Platonism

- Artificial human constructs - fantasy (?) consciously formulated and accepted

Bolzano´s Theo-Platonism:

The concept of SET (die Menge) consisting of elements, there can be infinite number of elements. „Proof" of existence of infinity: **God´s omniscience** – Infinite realm of truths.

Various sizes of infinite sets – longer segment of line bigger infinity of points…

Between all infinities can be found one-to-one (1-1) correspondence!

Cantor`s Theo-Platonism:

Criterion of equivalence (1-1) correspondence

(axiom-definition) x Bolzano

→ *Bolzano view is not correct (no alternative)*

Cantor`s theorem: more infinite numbers, also guaranteed by axioms.

Georg Cantor (1845-1916)

Cantor`s initial hesitation

Cantor came to „bigger" (uncountable) infinity: Quantity of limits of fundamental sequences of rational numbers – „real" numbers.

But: most of these „reals" are impossible to calculate! („mathematical jokes" – É. Borel) The amount of constructible sequences is the same as the amount of rational numbers! 1891 diagonal „proof" (numbers with undescribable decimal expansion)

Hidden voice of unconscious pragmatism

Cantor`s way to Platonism

After some hesitance, Cantor gave in to the temptation of traditional mathematical religion – theo-Platonism and accepted such "jokes" (i.e. supernatural axioms). So he got to the greater (uncountable) infinity.

→ Bolzano`s conception is not correct → no alternative concepts of infinity!

Cantor`s theorem:

$$P(X) > X$$

for infinite set axiom

Infinite „Tower of Babel"

20th cent. more effective methods to "find" greater and greater sizes of infinities. (How far can you count?)
Finding? Study? - In reality construction, the main tool - the acceptance of newer and more „supernatural" axioms.
These axioms define the existence of something that can't be positively defined. *E.g. we suppose that there is a "bigger" infinity of numbers, which have a quite irregular decimal expansion, but there is none that can be found!*
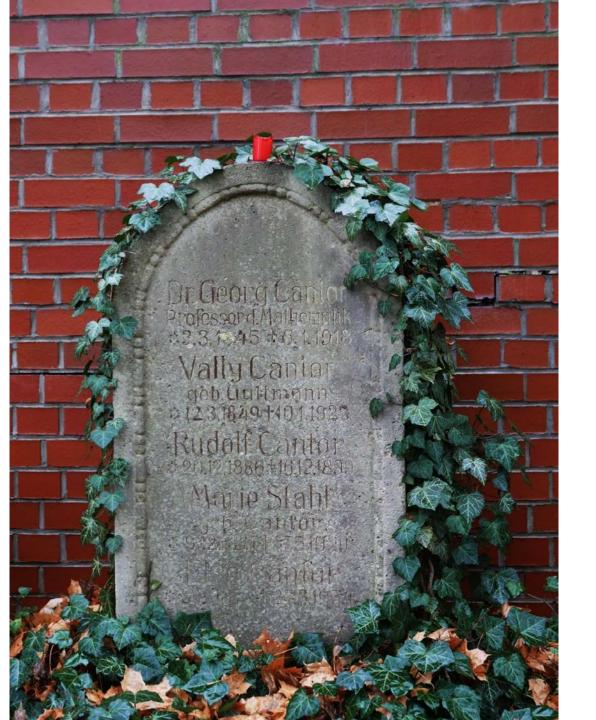
„Mathematical paradise from which no one will banish mathematicians any more",

David Hilbert

After initial hesitations and strong rejections, Cantor's theory was accepted. Cantor´s infinities were believed to be "facts", no alternativity was acceptable.

The whole mathematics could be based on the Set theory – Bourbaki group. Mathematicians found themselves in a one-way street.

Cantor's Paradise was a mysterious place, and mysteries attract.

Grave of Georg Cantor in Halle am Salle

Alternative approaches?

Intuitionism and Bolshevik menace

Luitzen Brouwer (1881–1966) rejected infinity as non-evident, **non-intuitive, and therefore unreal**.

The identification of non-evidence with no-reality – an axiom:

> *„Infinity is too big to be evident, i.e. to be real, to exist.”*

At the begining Anti-Platonic approach, mathematics – human invention, infinities (Bolzano`s, Cantor`s) are pseudoconcepts....

Later back to Platonism. No long life, so idiosyncratic and unsystematic, rejected as a "Bolshevik menace" (Frank Ramsey)

Petr Vopěnka in 2012

Vopěnka´s "visionary rule"

Cantorean infinities played no role in the world of phenomena and the world of physics:

The Cantorean view belonged among school prejudices.

Axiom :

The more elements some set contains, the lower is their value, their reality.

General collapse (return to Bolzano?)

The one-to-one correspondence can be found between all infinite sets.

particular application of "Voněnka´s visionary rule": the reality of higher infinities is even null.

Objections

Cantor demonstrated, there wasn't (1-1) correspondence between the set of integers and the set of real numbers!

If we accept only numbers that have any meaning in mathematics, then the set of them is countable. The uncountable rest of the "reals" are numbers without any meaning, they cannot be expressed by any means. **Cantor didn't find them, he didn't "discover" them, he even didn't construct them. He only defined that they should exist!** And it is possible to run mathematics without them (although some theorems would have to be adjusted).

And higher infinities, higher cardinals also play no role in practical mathematics and physics (Penrose). Mathematical jokes!

Vopěnka's journey to alternativity

Alternative Set Theory (AST)

<u>Why not to found mathematics</u> on something closer to the world of phenomena? **God, whether He exists or not, does not play any role in mathematics** and so do not "supernatural axioms". AST without "Babylonian Tower", without God hidden in "supernatural axioms", only two "sizes" of infinity. Alternative or Parallel concepts of infinity could be accepted.

Vopěnka believed, that there would be two theories, Cantorian and his "Alternative".

More "alternative" set theories had emerged, including those that didn't accept infinite sets. (Alexander Jesenin-Volpin, Doron Zeilberger, Toru Tsujishita, Edward Nelson)

Alternative math?

Is there possible coexistence of more set theories, alternative mathematics, all being right?

Vopěnka: yes, they are. Vopěnka abandoned Theo-Platonism on behalf of the realism of the world of phenomena. Mathematical entities represent our abstractions (subtraction of irrelevant aspects) from the phenomenal world. However, abstractions are possible to accomplish in many ways, the irrelevance of aspects depends on our point of view. So, there is **space for alternativity (plurality) in mathematics.**

More alternative approaches?
Without sets?

There could be very different mathematical foundations as well as different details if the results needed for the real world could be based on them.

Richard Hamming

In recent years, there has been a increasing belief that we could easily do without sets; that postulating such entities – such as sets, and sets of sets, etc. – is at least suspicious and could be only language artefacts. ... In addition, there are some "sociological" indicators: the purely mathematical exploration of set theory doesn't seem to be supported in the United States anymore. And perhaps once again, everything could return back to Prague (where the sets are still being studied), from which it came as a "wonderful flower of baroque". And possibly it could only turn out as merely" scholastic" research in a pejorative sense.

Jiří Fiala (1939-2012)

V posledních letech pozvolna sílí přesvědčení, že bychom se bez množin mohli klidně obejít, že postulování takových jsoucen, jako jsou množiny a množiny množin atd., je přinejmenším podezřelé a že může jít o jazykové artefakty.... Navíc jsou zde i některé ukazatele „sociologické": zdá se, že čistě matematická zkoumání teorie množin nenacházejí už ve Spojených státech podporu a že se možná jednou zase všechno vrátí do Prahy (kde se množiny ještě zkoumají), z níž to jako podivuhodný květ baroka vzešlo. A že se třeba ukáže, že to byla skutečně bádání (v pejorativním smyslu) školní, „scholastická".

Jiří Fiala

Vopěnka's return to Platonism
his catastrophic vision
the New Theory

Vopěnka gradually felt that the existence of more set theories was unacceptable; only one could be the right. At the end of his life, he came to decision that the only right theory was his own. He stopped using the name the "Alternative":

Today, this initially "Alternative" set theory is now a "New" set theory, because there exists no alternative.

And because traditional mathematics is built on a poor foundation, its whole building should be rebuilt!

In a nutshell, mathematical research has to start again from the point where it ended, and that is – at the beginning of the twentieth century...

Collapse of mathematics?

Mathematicians did not accept this catastrophic conclusion. The old mathematics worked well, so why talking about a collapse?

But, **why Vopěnka abandon his most beloved child**, the idea of alternativity? Why did he return to Platonism?

Older people subconsciously feel that they are losing the ground under their feet; their daily world is gradually disappearing. They are searching for something stable, looking for certainty. And the certainty can be offered by religion, and also by "mathematical religion", i.e. Platonism.

Moreover:

Vopěnka in his "alternative period": the classical set theory didn't represent any real foundation of math, it was merely a formal foundation – „decorative superstructure".

His criticism was directed at Cantor´s theory. However, isn´t it right also for Vopěnka´s alternativity itself? This was most likely what Vopěnka felt inside, although he didn't have enough courage to admit it. He lost the Cantorean paradise and now he did not want to loose the rest of his „mathematical religion".

The solution to the problem of alternativity

Behind all our consideration, there is subconscious "philosophy", which forms our paradigm, a framework of our considerations. („Psychoanalysis of maths") For progress in science, it is needed from time to time revise the philosophical background, and to introduce a new paradigm. (Kuhn)

Platonism – somewhat outdated paradigm, a myth. Mathematics was derived from counting, calculations and measuring of the world of phenomena. Mathematical objects and conceptions are our creations, and we can build them in various ways. Different basic concepts lead to different results, „truths".

Phenomena and interphenomena
(Hans Reichenbach)

Phenomena

One part of mathematical objects – phenomena (not very big numbers, shapes, simple relations, good old „natural" axioms, etc.). We create them automatically-unconsciously by interpreting data from our senses. This interpretation depends on our schemes of consideration, our hidden „philosophy", our prejudices. **The results of calculations that belong among phenomena must not contradict our experience.** There is no space for alternativity. Two plus two must be four.

Interphenomena

During the development of science (mathematics) there were also introduced "higher", non-phenomenal objects, the infinity first of all – interphenomena. They do not correspond to anything directly observable; they are our entirely consciously created constructs. We created them intending to go between and behind phenomena, to connect them to causal links, to complete and arrange them into the consistent system, i.e., to explain them. In the process of the creation, we use conscious presumptions.

The world of phenomena can be arranged in numerous ways, so interphenomena can also be constructed in many alternative ways. So, in the realm of higher-interphenomenal maths, we can tolerate alternative theories, various concepts of infinity, alternative truths.

Vopěnka's funeral of the alternativity of the set theories was premature.

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