Gödel Logics
Enduring Consequences of a short paper

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(joint work with Norbert Preining)

Kurt Gödel *Zum intuitionistischen Aussagenkalkül*, Anzeiger der Akademie der Wissenschaften Wien 69:65–66 (1952)

Kurt Gödel

Akademie der Wissenschaften in Wien

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schen Beziehungen entwickelt.

Das korr. Mitglied H. Hahn übersendet folgende Mitteilung:

»Zum intuitionistischen Aussagenkalkül« von Kurt Gödel in Wien.

Für das von A. Heyting[1] aufgestellte System H des intuitionistischen Aussagenkalküls gelten folgende Sätze:

I. *Es gibt keine Realisierung mit* endlich *vielen Elementen (Wahrheitswerten), für welche die und nur die in H beweisbaren Formeln erfüllt sind (d. h. bei beliebiger Einsetzung ausgezeichnete Werte ergeben).*

II. *Zwischen H und dem System A des gewöhnlichen Aussagenkalküls liegen unendlich viele Systeme, d. h. es gibt eine monoton abnehmende Folge von Systemen, welche sämtlich H umfassen und in A enthalten sind.*

Der Beweis ergibt sich aus folgenden Tatsachen: Sei F_n die Formel:

$$\sum_{1 \leqq i < k \leqq n} (a_i \supset\subset a_k)$$

wobei $\sum$ die iterierte $\vee$-Verknüpfung bedeutet und die a_i Aussagevariable sind. F_n ist erfüllt für jede Realisierung mit weniger als n Elementen, für welche alle in H beweisbaren Formeln erfüllt sind. Denn bei jeder Einsetzung wird in mindestens einem Summanden von F_n a_i und a_k durch dasselbe Element e ersetzt und $e \supset\subset e . \vee b$ ergibt bei beliebigem b einen ausgezeichneten Wert, weil die Formel $a \supset\subset a . \vee b$ in H beweisbar ist. Sei ferner S_n die folgende Realisierung:

Elemente: $\{1, 2 \ldots n\}$, ausgezeichnetes Element: 1;

$a \vee b = \min(a, b)$; $a \wedge b = \max(a, b)$; $a \supset b = 1$ für $a \geqq b$;
$a \supset b = b$ für $a < b$; $\neg\, a = n$ für $a \neq n$, $\neg\, n = 1$.

Dann sind für S_n sämtliche Formeln aus H und die Formel F_{n+1} sowie alle F_i mit größerem Index erfüllt, dagegen F_n sowie alle F_i mit kleinerem Index nicht erfüllt. Insbesonders ergibt sich daraus, daß kein F_n in H beweisbar ist. Es gilt übrigens ganz allgemein, daß eine Formel der Gestalt $A \vee B$ in H nur dann beweisbar sein kann, wenn entweder A oder B in H beweisbar ist.

Let $V \subseteq [0,1]$ be some set of truth values which contains 0 and 1. A propositional Gödel valuation $\mathcal{I}^0$ (short valuation) based on V is a function from the set of propositional variables into V with $\mathcal{I}^0(\bot) = 0$. This valuation can be extended to a function mapping formulas from $\mathrm{Frm}(\mathscr{L}^0)$ into V as follows:

$$\mathcal{I}^0(A \wedge B) = \min\{\mathcal{I}^0(A), \mathcal{I}^0(B)\},$$

$$\mathcal{I}^0(A \vee B) = \max\{\mathcal{I}^0(A), \mathcal{I}^0(B)\},$$

$$\mathcal{I}^0(\triangle A) = \begin{cases} 1 & \mathcal{I}^0(A) = 1, \\ 0 & \mathcal{I}^0(A) < 1, \end{cases}$$

$$\mathcal{I}^0(A \to B) = \begin{cases} \mathcal{I}^0(B) & \text{if } \mathcal{I}^0(A) > \mathcal{I}^0(B), \\ 1 & \text{if } \mathcal{I}^0(A) \le \mathcal{I}^0(B). \end{cases}$$

A formula is called valid with respect to V if it is mapped to 1 for all valuations based on V. The set of all formulas which are valid with respect to V will be called the propositional Gödel logic based on V and will be denoted by $\mathbf{G}^0_V$.
The validity of a formula A with respect to V will be denoted by

$$\models^0_V A \quad \text{or} \quad \models_{\mathbf{G}^0_V} A.$$

Let $\neg A$ be $A \to \bot$ and $A \prec B$ be $(B \to A) \to B$.

$$\mathcal{I}^0(\neg A) = \begin{cases} 0 & \text{if } \mathcal{I}^0(A) > 0, \\ 1 & \text{otherwise,} \end{cases}$$

$$\mathcal{I}^0(A \prec B) = \begin{cases} 1 & \text{if } \mathcal{I}^0(A) < \mathcal{I}^0(B) \text{ or } \mathcal{I}^0(A) = \mathcal{I}^0(B) = 1, \\ \mathcal{I}(B) & \text{otherwise.} \end{cases}$$

We assume closed V and countable Γ. If Γ is a set of formulas (possibly infinite), we say that Γ *entails* A *in* $\mathbf{G}_V$, $\Gamma \models_V A$ iff for all $\mathcal{I}$ into V, $\mathcal{I}(\Gamma) \leq \mathcal{I}(A)$.
Γ 1-*entails* A in $\mathbf{G}_V$, $\Gamma \rightarrow_V A$, iff, for all $\mathcal{I}$ into V, whenever $\mathcal{I}(B) = 1$ for all $B \in \Gamma$, then $\mathcal{I}(A) = 1$.

Proposition

$\Pi \models_V A$ iff $\Pi \rightarrow_V A$.

Examples

$$\models (A \to B) \lor (B \to A)$$

$$\models (A \to B) \lor ((A \to B) \to A)$$

$$\models \neg A \lor \neg\neg A$$

$$\models A \to B \lor B \to C \lor C \to D$$

Let $G_V = \{A : \models_{G_V} A\}$ be the propositional Gödel logic for V.

Proposition

(i) $G_V = G_{V'}$ iff $|V| = |V'|$ or both V, V' are infinite
(ii) $G_V \subsetneq G_{V'}$ iff $|V| < |V'|$
(iii) $\bigcap_{|V| finite} G_V = G_{[0,1]}$
(iv) Assume A contains n variables, then

$$G_{n+2} \models A \Rightarrow \text{ for all } V\text{: } G_V \models A$$

Proof

i, iii, iv are obvious
ad ii. $G_V \subseteq G_{V'}$ is obvious, and $G_V \models A_{|V|}$, but $G_{V'} \not\models A_{|V|}$ for

$$A_{|V|} = p_1 \vee p_1 \rightarrow p_2 \vee \ldots \vee p_{|V|} \rightarrow \top$$

Gödel Conditional

Suppose we have a standard language containing a 'conditional' $\twoheadrightarrow$ interpreted by a truth-function into $[0, 1]$, and some entailment relation $\models$. Suppose further that

> a conditional evaluates to 1 if the truth value of the antecedent is less or equal to the truth value of the consequent, i.e., if $\mathcal{I}(A) \leq \mathcal{I}(B)$, then $\mathcal{I}(A \twoheadrightarrow B) = 1$;
>
> if $\Gamma \models B$, then $\mathcal{I}(\Gamma) \leq \mathcal{I}(B)$;
>
> the deduction theorem holds, i.e., $\Gamma \cup \{A\} \models B \Leftrightarrow \Gamma \models A \twoheadrightarrow B$.

Then $\twoheadrightarrow$ is the Gödel conditional.

Proof

From (1), we have that $\mathcal{I}(A \twoheadrightarrow B) = 1$ if $\mathcal{I}(A) \leq \mathcal{I}(B)$. Since $\models$ is reflexive, $B \models B$. Since it is monotonic, $B, A \models B$. By the deduction theorem, $B \models A \twoheadrightarrow B$. By (2),

$$\mathcal{I}(B) \leq \mathcal{I}(A \twoheadrightarrow B).$$

From $A \twoheadrightarrow B \models A \twoheadrightarrow B$ and the deduction theorem, we get $A \twoheadrightarrow B, A \models B$. By (2),

$$\min\{\mathcal{I}(A \twoheadrightarrow B), \mathcal{I}(A)\} \leq \mathcal{I}(B).$$

Thus, if $\mathcal{I}(A) > \mathcal{I}(B)$, $\mathcal{I}(A \twoheadrightarrow B) \leq \mathcal{I}(B)$.

Theorem

(i) $\models_V$ is compact iff V is uncountable
(ii) There are uncountably many different $\{< \Gamma, A >: \Gamma \models_V A\}$

Example: $V = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ does not admit a compact entailment
Let $\Gamma = \{x_1 < x_2, x_2 < x_3, \ldots\} \cup \{x_1 > z, x_2 > z, \ldots\}$
$\Gamma \models_V z$ but $\Gamma' \not\models z$ for all finite subsets $\Gamma' \subset \Gamma$.

Axioms and deduction systems for Gödel logics

We will denote by **IL** the following complete axiom system for intuitionistic logic.

I1	$\bot \to A$	I8	$(A \to B) \to [(C \to A) \to (C \to B)]$
I2	$A \to (B \to A)$	I9	$[A \to (C \to B)] \to [C \to (A \to B)]$
I3	$(A \wedge B) \to A$	I10	$(A \to C) \wedge (B \to C) \to ((A \vee B) \to C$
I4	$(A \wedge B) \to B$	I11	$(C \to A) \wedge (C \to B) \to (C \to (A \wedge B)$
I5	$A \to (B \to (A \wedge B))$	I12	$(A \to (B \to C)) \to (A \wedge B \to C)$
I6	$A \to (A \vee B)$	I13	$[A \to (A \to B)] \to (A \to B)$
I7	$B \to (A \vee B)$		
MP	$\dfrac{A \quad A \to B}{B}$		

Theorem

$\mathbf{G}_{[0,1]}$ is axiomatized by $\mathrm{IL} + (A \to B \vee B \to A)$

Proof

A chain on $X_1, \ldots, x_n$ is an expression

$$(\bot \bowtie_0 x_{\pi(1)}) \wedge (x_{\pi(1)} \bowtie_1 x_{\pi(2)}) \wedge \ldots \wedge (x_{\pi(n)} \bowtie_n \top)$$

where π is a permutation and $\bowtie_i \in \{\prec, \to\}$.
$\bigvee_{\substack{C chain \\ \text{on } \{x_1,\ldots,x_n\}}} C$ is valid (use that all Gödel logics prove $\models_{[0,1]} u \prec v \vee u \leftrightarrow v \vee v \prec u$).

Proof cont.

Let $\mathcal{F}(x_1, \ldots, x_n)$ be the set of formulas in $x_1, \ldots, x_n$, $\psi_C : \mathcal{F}(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n, \top, \bot\}$ the formal evaluation of a formula under C, then

$$C \wedge A \leftrightarrow C \wedge \psi_c(A)$$

A formula is valid iff $\psi_C(A) = 1$ for all C.

$$\bigvee C \leftrightarrow \bigvee C \wedge \top \leftrightarrow \bigvee (C \wedge \psi_C(A)) \leftrightarrow \bigvee (C \wedge A) \leftrightarrow (\bigvee C) \wedge A \leftrightarrow A$$

Corollary

Strong completeness for uncountable V follows from compactness.

Corollary

$\mathbf{G}_{|V|}$ with $|V| = n$ is axiomatize by

$$\mathbf{G}_{[0,1]} + \top \to A_1 \vee A_1 \to A_2 \vee \ldots \vee A_{n-1} \to \bot$$

Gödel logics with $\triangle$

$$v(\triangle A) = \begin{cases} 1 & \text{if } v(A) = 1 \\ 0 & \text{if } v(A) \neq 1 \end{cases}$$

Theorem

$\mathbf{G}_{[0,1]}$ extended by $\triangle$ is axiomatized by $\mathbf{G}_{[0,1]}$ and

$$\begin{array}{ll} \Delta 1 & \triangle A \vee \triangle A \\ \Delta 2 & \triangle(A \vee B) \rightarrow (\triangle A \vee \triangle B) \\ \Delta 3 & \triangle A \rightarrow A \\ \Delta 4 & \triangle A \rightarrow \triangle\triangle A \\ \Delta 5 & \triangle(A \rightarrow B) \rightarrow (\triangle A \rightarrow \triangle B) \\ \Delta 6 & \dfrac{A}{\triangle A} \end{array}$$

First order Gödel Logics

Definition

A *Gödel set* is a closed set $V \subseteq [0,1]$ which contains 0 and 1.
Let V be a Gödel set. An *interpretation* $\mathcal{I}$ *into* V, or a *V-interpretation*, consists of

- a nonempty set $U = U^{\mathcal{I}}$, the 'universe' of $\mathcal{I}$,
- for each k-ary predicate symbol P, a function $P^{\mathcal{I}}\colon U^k \to V$,
- for each k-ary function symbol f, a function $f^{\mathcal{I}}\colon U^k \to U$,
- for each variable v, a value $v^{\mathcal{I}} \in U$.

Given an interpretation $\mathcal{I}$, we can naturally define a value $t^{\mathcal{I}}$ for any term t and a truth value $\mathcal{I}(A)$ for any formula A of $\mathscr{L}^U$. For a term $t = f(u_1, \ldots, u_k)$ we define $\mathcal{I}(t) = f^{\mathcal{I}}(u_1^{\mathcal{I}}, \ldots, u_k^{\mathcal{I}})$. For atomic formulas $A \equiv P(t_1, \ldots, t_n)$, we define $\mathcal{I}(A) = P^{\mathcal{I}}(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}})$. For composite formulas A we extend the truth definitions from the propositional case for the new syntactic elements by:

$$\mathcal{I}(\forall x\, A(x)) = \inf\{\mathcal{I}(A(u)) : u \in U\}$$
$$\mathcal{I}(\exists x\, A(x)) = \sup\{\mathcal{I}(A(u)) : u \in U\}.$$

If $\mathcal{I}(A) = 1$, we say that $\mathcal{I}$ *satisfies* A, and write $\mathcal{I} \models A$. If $\mathcal{I}(A) = 1$ for every V-interpretation $\mathcal{I}$, we say A is *valid* in $\mathbf{G}_V$ and write $\mathbf{G}_V \models A$.

$$V_{\mathbb{R}} = [0,1] \qquad V_0 = \{0\} \cup [1/2, 1]$$
$$V_{\downarrow} = \{1/k \mid k \geq 1\} \cup \{0\}$$
$$V_{\uparrow} = \{1 - 1/k \mid k \geq 1\} \cup \{1\}$$
$$V_n = \{1 - 1/k \mid 1 \leq k \leq m - 1\} \cup \{1\}$$

The corresponding Gödel logics are $\mathbf{G}_{[0,1]}$, $\mathbf{G}_0$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, and $\mathbf{G}_n$. $\mathbf{G}_{[0,1]}$ is the *standard* Gödel logic.

Theorem

$$\mathbf{G}_{\uparrow} = \bigcap_{V : |V| \text{ is finite}} \mathbf{G}_V$$
$$\mathbf{G}_{[0,1]} = \bigcap_{\text{all } V} \mathbf{G}_V$$

$$\mathbf{G}_n \supsetneq \mathbf{G}_{n+1},$$

$$\mathbf{G}_n \supsetneq \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\mathbb{R},$$

$$\mathbf{G}_n \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R},$$

$$\mathbf{G}_0 \supsetneq \mathbf{G}_\mathbb{R}.$$

$$\mathbf{G}_n \supsetneq \bigcap_n \mathbf{G}_n = \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_{[0,1]} = \bigcap_V \mathbf{G}_V.$$

Intuitionistic First Order Logic $\mathbf{IL}^1$ extends $\mathbf{IL}$ by

$$\frac{A \to B(a)}{A \to \forall x B(x)} \qquad \forall x B(x) \to B(t)$$

$$\frac{A(a) \to B}{\exists x A(x) \to B} \qquad A(t) \to \exists x A(x)$$

(a does not occur in the lower sequent)

Axiomatizability results

Axiomatizable case 1: 0 is contained in the perfect kernel

$\mathbf{G}_V$ is axiomatized by

$$\mathbf{IL} \quad + \quad A \to B \vee B \to A \quad + \quad \forall x(A \vee B(x)) \to A \vee \forall x B(x)$$

Remark: $\mathbf{G}_V = \mathbf{G}_{V'}$ iff V, V' are uncountable and 0 is in the perfect kernel of each of them.

Axiomatizable case 2: 0 is isolated

$\mathbf{G}_V$ is axiomatized by

$$\mathbf{G}_{[0,1]} \quad + \quad \forall \bar{y}(\neg \forall x A(x, \bar{y}) \to \exists x \neg A(x, \bar{y}))$$

Remark: $\mathbf{G}_V = \mathbf{G}_{V'}$ if both are uncountable with 0 isolated.

Axiomatizable case 3: Finite Gödel sets

$\mathbf{G}_V$ with $|V| = n$ is axiomatized by

$$\mathbf{G}_{[0,1]} \quad + \quad \top \to A_1 \vee A_1 \to A_2 \vee \ldots \vee A_{n-1} \to \bot$$

Not recursively enumerable case 1: Countable Gödel sets

Let $A^g \equiv$

$$\left\{ \begin{array}{l} S \wedge c_1 \in 0 \wedge c_2 \in 0 \wedge c_2 \prec c_1 \wedge \\ \quad \forall i \left[\forall x, y \forall j \forall k \exists z \, D \vee \forall x \neg (x \in s(i)) \right] \end{array} \right\} \to (A' \vee \exists u \, P(u))$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$D \equiv \begin{array}{c} (j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y) \to \\ \to (z \in s(i) \wedge x \prec z \wedge z \prec y) \end{array}$$

Not recursively enumerable case 2: 0 not isolated but not in the perfect kernel

Let $A^h \equiv$

$$\left\{\begin{array}{l} S \wedge \forall n((Q(n) \to Q(s(n))) \to Q(n)) \wedge \\ \quad \neg\forall n\, Q(n) \wedge \forall n \neg\neg Q(n) \wedge \\ \quad \forall n \forall x((Q(n) \to P(x,n)) \to Q(n)) \wedge \\ \quad \forall n \exists x \exists y (x \in_n 0 \wedge y \in_n 0 \wedge x \prec_n y) \wedge \\ \quad\quad \forall n \forall i \left[\forall x, y \forall j \forall k \exists z\, E \vee \forall x \neg(x \in_n s(i))\right] \end{array}\right\} \to (A' \vee \exists n \exists u\, P(u,n) \vee \exists$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$E \equiv \begin{array}{c} (j \leq i \wedge x \in_n j \wedge k \leq i \wedge y \in_n k \wedge x \prec_n y) \to \\ \to (z \in_n s(i) \wedge x \prec_n z \wedge z \prec_n y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(n) \equiv \forall i \exists x (x \in_n i)$.

Relation to Kripke frames

Theorem

For every countable linear Kripke frame K there is a Gödel set V_K such that $\mathbf{L}(K) = \mathbf{G}_{V_K}$.

Theorem

The set of infinitely-valued propositional Gödel logics is singleton.
The set of infinitely-valued first-order Gödel logics is countable.
The set of infinitely-valued propositional and first-order entailments is uncountable.
The set of infinitely-valued propositional Gödel logics with propositional quantifiers is uncountable.

Theorem

For every n there is exactly one n-valued propositional logic, n-valued propositional logic with quantifiers, n-valued first-order logic, n-valued first-order logic with entailment.

Gödel, Kripke frames and Intuitionistic Logic

Gödel (1933)

Wanted to show that Intuitionistic Logic does not have a finite matrix, i.e., is not a finitely valued logic.

Kripke (60ies)

Semantic for Intuitionistic Logic based on trees.
Axiom $(A \to B) \vee (B \to A)$ of Gödel logics implies linearity on Kripke frames.

Relating Gödel logics and logic on Kripke frames

'Truth values in Kripke frames'

Sets of worlds in which a formula is true, is upward closed.
The set of upwards closed sets in K, $\mathrm{Up}(K)$, is a Gödel algebra.
A (order theoretic) upper limit point w generates two distinct upward closed sets:

$$w^{\uparrow} = \{v \in K : R(w,v)\}$$
$$w^{\uparrow *} = w^{\uparrow} \setminus \{w\}$$

The logic $\mathbf{L}(\mathbb{Q})$ cont.

An embedding of $\mathbb{Q}'$ into $[0, 1]$ preserving the order, infima and suprema will generate a set which is isomorph to the border points of the Cantor middle third set. The closure of this set is the Cantor middle third set.

The logic $\mathbf{L}(\mathbb{Q})$ cont.

An embedding of $\mathbb{Q}'$ into $[0,1]$ preserving the order, infima and suprema will generate a set which is isomorph to the border points of the Cantor middle third set. The closure of this set is the Cantor middle third set.

Thus, $\mathbf{L}(\mathbb{Q}) = \mathbf{G}_{\mathbb{C}_{[0,1]}} = \mathbf{G}_{[0,1]}$

Equivalence result

Gödel logic to Kripke frame

For each Gödel logic there is a countable linear Kripke frame such that the respective logics coincide.

Kripke frames to Gödel logic

For each countable linear Kripke frame there is a Gödel truth value set such that the respective logics coincide.

Definition (sequent)

A sequent is

$$\Gamma \vdash \Delta$$

where Γ, Δ are multisets of formulas and $|\Delta| \leq 1$.

Sequent calculus **LJ** - structural rules

Axiom

$A \vdash A$

weakening

$$\frac{\Gamma \vdash \Delta}{A_1, \Gamma \vdash \Delta} \, w_l \qquad \frac{\Gamma \vdash}{\Gamma \vdash A} \, w_r$$

contraction

$$\frac{A_1, A_1, \Gamma \vdash \Delta}{A_1, \Gamma \vdash \Delta} \, c_l$$

cut

$$\frac{\Gamma \vdash A \qquad A, \Pi \vdash \Delta}{\Gamma, \Pi \vdash \Delta} \, cut(A)$$

$|\Delta| \leq 1$

Sequent calculus **LJ** - logical rules

and $\wedge$

$$\frac{A_1, \Gamma \vdash \Delta}{A_1 \wedge A_2, \Gamma \vdash \Delta} \wedge_{l_1} \quad \frac{A_2, \Gamma \vdash \Delta}{A_1 \wedge A_2, \Gamma \vdash \Delta} \wedge_{l_2} \quad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_r$$

or $\vee$

$$\frac{A_1, \Gamma \vdash \Delta \qquad A_2, \Gamma \vdash \Delta}{A_1 \vee A_2, \Gamma \vdash \Delta} \vee_l \quad \frac{\Gamma \vdash A_1}{\Gamma \vdash A_1 \vee A_2} \vee_{r_1} \quad \frac{\Gamma \vdash A_2}{\Gamma \vdash A_1 \vee A_2} \vee_{r_2}$$

not $\neg$

$$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash} \neg_l \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \neg_r$$

implication $\rightarrow$

$$\frac{\Gamma \vdash A_1 \qquad A_2, \Gamma \vdash \Delta}{A_1 \rightarrow A_2, \Gamma \vdash \Delta} \rightarrow_l \qquad \frac{A_1, \Gamma \vdash A_2}{\Gamma \vdash A_1 \rightarrow A_2} \rightarrow_r$$

$|\Delta| \leq 1$

Sequent calculus **LJ** - logical rules

for all $\forall$

$$\frac{A\{x \leftarrow t\}, \Gamma \vdash \Delta}{(\forall x)A(x), \Gamma \vdash \Delta}\,\forall_l \qquad \frac{\Gamma \vdash A\{x \leftarrow \alpha\}}{\Gamma \vdash (\forall x)A(x)}\,\forall_r$$

t term, does not contain any variables which are bound in A and α is a free variable which may not occur in Γ, Δ, A. α is called an eigenvariable.

there exists $\exists$

$$\frac{A\{x \leftarrow \alpha\}, \Gamma \vdash \Delta}{(\exists x)A(x), \Gamma \vdash \Delta}\,\exists_l \qquad \frac{\Gamma \vdash A\{x \leftarrow t\}}{\Gamma \vdash (\exists x)A(x)}\,\exists_r$$

The variable conditions for $\exists_l$ are the same as those for $\forall_r$ and similarly for $\exists_r$ and $\forall_l$.

$|\Delta| \leq 1$

Definition (hypersequent)

A hypersequent is a multiset

$$\Gamma_1 \vdash A_1 \mid \ldots \mid \Gamma_n \vdash A_n$$

where for every $i = 1, \ldots, n$, $\Gamma_i \vdash A_i$ is a sequent, called component of the hypersequent.

Axioms

$A \vdash A \qquad \bot \vdash$

A is atomic

Cut Rule

$$\frac{G \mid \Gamma' \vdash A \quad G' \mid A, \Gamma \vdash C}{G \mid G' \mid \Gamma, \Gamma' \vdash C} \ (cut)$$

External Structural Rules

$$\frac{G}{G \mid \Gamma \vdash A} \ (ew)$$

Internal Structural Rules

$$\frac{G \mid \Gamma \vdash C}{G \mid \Gamma, A \vdash C} \ (w, l) \qquad \frac{G \mid \Gamma \vdash}{G \mid \Gamma \vdash C} \ (w, r)$$

$$\frac{G \mid \Gamma, \Gamma' \vdash A \quad G' \mid \Gamma_1, \Gamma_1' \vdash A'}{G \mid G' \mid \Gamma, \Gamma_1' \vdash A \mid \Gamma', \Gamma_1 \vdash A'} \ (com)$$

Logical Rules $\dfrac{G \mid S}{G \mid S''}$ $\dfrac{G \mid S \qquad G \mid S'}{G \mid S''}$

for S, S', S'' as in the logical rules for **LJ**.

Theorem

If $d \vdash H$, one can find a cut-free proof $d' \vdash H$ with $|d'| \leq 4^{|d|}_{\rho(d)}$.

Corollary (Midhypersequent theorem)

For every valid hypersequent of prenex formulas there exists a hypersequent (the midhypersequent) such that all inferences in the proof above are propositional or structural and all inferences below are quantificational or structural.

Corollary

The prenex fragment of $\mathbf{G}_{[0,1]}$ *admits Skolemization.*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \vdash B \mid B \vdash A}}{A \vdash B \mid \ \vdash B \to A}}{\vdash A \to B \mid \ \vdash B \to A}}{\vdash A \to B \vee B \to A \mid \ \vdash B \to A}}{\vdash A \to B \vee B \to A \mid \ \vdash A \to B \vee B \to A}}{\vdash A \to B \vee B \to A}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{B(a) \vdash B(a) \qquad A \vdash A}{B(a) \vdash A \mid A \vdash B(a)}}{A \lor B(a) \vdash A \mid A \vdash B(a)} \qquad B(a) \vdash B(a)}{A \lor B(a) \vdash A \mid A \lor B(a) \vdash B(a)}}{\forall x(A \lor B(x)) \vdash A \mid A \lor B(a) \vdash B(a)}}{\forall x(A \lor B(x)) \vdash A \mid \forall x(A \lor B(x)) \vdash B(a)}}{\forall x(A \lor B(x)) \vdash A \mid \forall x(A \lor B(x)) \vdash \forall x B(x)}}{\forall x(A \lor B(x)) \vdash A \lor \forall x B(x) \mid \forall x(A \lor B(x)) \vdash \forall x B(x)}}{\forall x(A \lor B(x)) \vdash A \lor \forall x B(x) \mid \forall x(A \lor B(x)) \vdash A \lor \forall x B(x)}}{\forall x(A \lor B(x)) \vdash A \lor \forall x B(x)}}{\vdash \forall x(A \lor B(x)) \to A \lor \forall x B(x)}$$

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