Petr Vopěnka – From Topology to Set Theory

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The topology knows more than three definition of dimension of a topological space X

$$ind(X),\ Ind(X) \text{ and } dim(X).$$

The small inductive dimension $ind(X)$ is defined by mathematical induction as follows:

$ind(\emptyset) = -1$,

$ind(X) \leq n$ if

$$(\forall x \in X)(\forall \text{ open } U \ni x)(\exists \text{ open } V \ni x)\,(V \subseteq U \wedge ind(\mathrm{Bd}(V)) \leq n-1).$$

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The covering dimension $dim(X)$ is defined as follows:
$dim(X) \leq n$ if for every open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$, refinement of $\mathcal{U}$, such that for every $\mathcal{V}_0 \subseteq \mathcal{V}$, $|\mathcal{V}_0| \geq n+2$ we have $\bigcap \mathcal{V}_0 = \emptyset$.

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Petr Vopěnka in his master thesis has proved

Theorem (Petr Vopěnka)

For any integers $0 \leq m < n$ there exist compact topological spaces X, Y such that

$$dim(X) = m,\ ind(X) = n,\ dim(Y) = m,\ Ind(Y) = n.$$

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He constructed a model of the Gödel-Bernays set theory by ultrapower and showed that this model is often non-well founded.

Then using well founded model constructed by a measurable cardinal, Petr Vopěnka presented a very nice proof of a recent result by Dana Scott, that the existence of a measurable cardinal is incompatible with the axiom of constructibility.

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Let me remind that during the session of the seminary, Petr Vopěnka usually sit down by the last table, of course with ashtray and cigarette, and let the session go without his interruption.

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Finally, in April 1964, Petr Vopěnka in his seminary let presented paper in which the independence of Continuum Hypotheses of the Gödel-Bernays set theory was proved. The topologist Petr Vopěnka considered some clopen subsets of the generalized Cantor space $C(\kappa)$ as forcing condition. Actually, he has already defined the "Boolean value" of a formula in this forcing as an open subset of $C(\kappa)$. The universe of the model was an ultraproduct of generalized Gödel's structure of the constructible universe by A. Lévy and A. Hajnal over $C(\kappa)$. The ultrafilter had to contain all open dense subsets of $C(\kappa)$. Full paper was published Russian in Commentationes Mathematicae Universitas Carolinae (shortly CMUC) in 1964 (English translation in AMS Translations 1966).

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Boolean-valued models are still the main tools of showing relative consistency of some sentences of the set theory.

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As a jock, Petr Vopěnka formulated today called "Vopěnka's Principle" which plays an important rôle in the theory of large cardinals.

Important rôle played a result with B. Balcar:

Theorem (Balcar - Vopěnka)

If two inner models M, N, one with the axiom of choice, have same sets of ordinals, then $M = N$.

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The simplest model of semisets:
Let $M \subseteq V$ be an inner model. Elements of M are sets, subsets of M are semisets and subclasses of M are classes.

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In 1972, Petr Vopěnka and Petr Hájek published the monograph The Theory of Semisets.

The monograph contains several important results of the set theory formulated in the language of semiset theory. The translation is easy. I present two important results of the monograph.

Let M be an inner model. A set $\sigma \subseteq M$ is said to be a support over M if for any two binary relations $r_1, r_2 \in M$, there exists a binary relation $r \in M$ such that $r"\sigma = r_1"\sigma \setminus r_2"\sigma$, where

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B. Balcar find a very nice proof of an strenghtening of this result.

Theorem (B. Balcar – P. Vopěnka)

If $\sigma \subseteq P \in M$ is a support, then there exists an preorder $\leq \in M$ of the set P such that σ is a filter on $\langle P, \leq \rangle$ generic over M.

If M is an inner model, Petr Vopěnka defined the boundedness axiom

$$Bd(\kappa) \equiv (\forall \sigma \subseteq M)(\exists a \in M, |a|^M < \kappa)(\exists \rho \subseteq a)\, \sigma = \bigcup \rho.$$

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I will not comment the very detailed style of presentation of this monograph. Just I recommend you to try to find and to understand the above mentioned results in the monograph.

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The Czech school's work in Set Theory in the 1960's and 70's is not readily accessible in the West. Personally, I found the paper both useful and interesting.

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Let me preset in the alphabetical order the most important members of the seminary and therefore Vopěnka's students: Bohuslav Balcar, L.B., Petr Hájek, Tomáš Jech, Karel Hrbáček, Karel Príkry (he studied in Warsaw), Antonín Sochor, Petr Štěpánek.

Thank You for Your Attention