Alternative Set Theory and New Infinitary Mathematics

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Origins of Alternative Set Theory (from around 1973)

Vopěnka's scepticism about Classical Set Theory

- Number of principles independent of the core axioms.
- Lack of intuition to decide between them.

Philosophical motivation

- Phenomenology (Husserl): focus on phenomena of the natural real world.
- Crucial concepts: horizon, indistinctness.

Intuition about Infinity

Our intuition about infinity comes from encountering large finite sets.

Intuition for the new theory

Countably infinite collections are those that can represent steps on a path to the horizon.

Horizon should be understood as a general limit for perception.

Paths to the horizon continue in the same manner at least a little further.

Finite collections are those that can be distinctly individually surveyed.

Infinite collections are those that are not finite (they contain countably infinite subcollections).

Non-standard Analysis

Abraham Robinson (1960, 1966) employed ideas from mathematical logic to develop analysis with infinitesimals.

He used a structure within classical set universe that may be seen as one mathematical world within another, larger one.

This was another encouragement for Vopenka - Alternative Set Theory formally resembles nonstandard analysis.

In spite of expectations infinitesimals did not come back to mainstream maths.

Vopenka hoped for the new theory to enable proper rehabilitation of infinitesimals.

Alternative Set Theory (1979 version)

Theory with sets and classes, with all sets being also classes but not vice versa.

Sets satisfy axioms that are equivalent to ZF with the negation of the Axiom of infinity.

A proper semiset is a class such that it is not a set but is contained in some set (as a subclass of it).

Axiom: There is a proper semiset.

A finite set (necessarily a set) is a set with no proper sub-semisets.

Alternative Set Theory (1979 version)

Natural numbers are defined as usual (von Neumann definition), they form N. The finite ones form FN.

$FN \subset N$. Finite numbers represent steps on a journey to the horizon.

Axiom of prolongation: For any function on FN there is a set function extending it.

Axiom of Two cardinalities: For any class that is not finite there is a (class-)bijection onto N or onto FN.

Axioms of Extensionality, Comprehension for Classes and Extensional Coding (Choice).

Witnessed and limit universe

1979 Mathematics in Alternative Set Theory aimed to describe what he called *limit universe* - where no particular natural number is infinite.

Vopěnka called also for investigation aimed at *witnessed universes* where there would be particular natural numbers that are not finite. Such a theory would be inconsistent in the classical sense.

Vopenka suggested that in this case, only proofs of finite lengths (from FN) should be accepted, but he has not pursued this further.

After Alternative Set Theory

Alkternative Set Theory continued develop to some extent after 1979, but remained relatively unknown.

For over 20 years, Vopenka devoted himself mainly to philosophy and history of mathematics, or politics.

By 2010, he returned to mathematics calling again for rethinking of the foundations. He died in 2015 and his later thoughts on mathematics are gathered in the four volumes of New Infinitary Mathematics.

New Infinitary Mathematics

The Grand Illusion Vopěnka surveys historical roots of the classical approach to infinity, notably Bolzano's proof of the existence an infinite set. Then he goes over the ultraproduct construction, focusing on natural numbers and argues that it shows the set of natural numbers does not exist.

(He assumes von Neumann's representation of natural numbers. If the set existed, the ultraproduct would yield ultraextension operator (that makes sets out of functions) and produce more natural numbers, along with another, longer set of natural numbers.)

New Infinitary Mathematics

New Theory of Sets and Semisets Vopěnka returns to the intuition that motivated Alternative Set Theory but remains quite informal.

" *We can use predicate calculus when studying elements of the semiset FN, their properties and relations. However, we must be very careful with the quantifiers because the semiset FN is not sharply defined in the direction of the horizon limiting the size of finite natural numbers."*

He dwells on reasons for introducing various concepts from before, and discusses the geometrical horizon and the Greek mathematical world (limited by the geometrical horizon). Mathematics is developed similarly to the 1979 book.

New Infinitary Mathematics

Real Numbers, Differential Calculus

Here the framework goes beyond phenomenologically motivated mathematics.

Two mathematical worlds: the Greek mathematical world and an extension of it related to the expansion of natural infinity to the absolute infinity of Bolzano and Cantor.

Particularly accessible and elegant axiomatic treatment of infinitesimal calculus (although the integral calculus part is not quite completed).

World-within-world framework, formally similar to other axiomatic approaches (based on nonstandard analysis).

From *The Harrowing Secret*

He was living this rarest moment of enlightenment that only a few chosen scholars encounter, possibly just once in their lives. It is a moment of indescribable excitement, marrying a religious thanks-giving with a solemn apprehension to take the fruit that has ripened. This fruit lies in front of us, attracts our eyes and at the same time repels the hand from reaching out to it. We know that we have it now, it is ours, but its glow dazzles us and holds us in a sweet intoxication so that we do not find enough strength to take it completely.

From *The Harrowing Secret*

And yet at the same time we long to become sober because we know that we must still make the last step. Never are we so frightened of death as at this very moment: only that can catch us and harvest the crown of our life, turn our moment of seeing into nothing. Yes, this moment truly appears to us as the peak of what a scholar can achieve. Whoever has lived through it once will not stop longing for it. And whoever, even though seeking, has never lived through it, is, without being aware of it, seeking that, and not what they imagine to be their goal.