Relativistic optics in gravitational lensing: The focus of a cluster and its aberrations

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In this article, general idea of focusing is studied within the framework of optics extension into general relativity (*covariant optics*). In a configuration of static spacetime, the general, mathematically rigorous treatment of rays, wavefronts and caustics of spherical symmetry is presented, particularly with regard to problems of obtaining them within general relativity. An original result is the aberration formulation to covariant optics, whose application is given in this paper; a particular solution of Einstein equations is finally chosen to provide concrete, exact results of cluster focal length and its aberration structure. In this way, a gravitational lensing situation is shown to be a true lens.

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Introduction

The gravitational lensing, as comprehensively described, *e.g.* in [1], is very successful in providing reasonable results, *e.g.* [2–4]. Recently, the trend of mathematically more sophisticated treatment appeared, see *e.g.* [5, 6]. Though very potent, most of these approaches are, however, only further approximations to relativistic optics — which is hereby defined as an extension of geometrical optics [7] to curved case, *i.e.* the covariant eikonal equation and lowest-order amplitude transfer covariant equation [8]. In this article, parts of aberrational formulation to relativistic optics (which is exact from the viewpoint of gravitational lensing) are studied.

Many principal ideas of classical optics are valid in curved spacetimes as well, however, there are some, that cannot be treated therein. Yet, some of the latter can be redefined in a generalised way, so that they are meaningful in curved cases and reduce to well established ones in the flat case. The idea of (positive) focusing is abstract enough in this sense, when requiring that adjacent rays touch (in mathematical sense). We show in this article, how this can be dealt with via caustic study within curved spacetimes.

We shall present the treatment in static, spherically symmetric case, allowing us — after choosing, *e.g.*, a point source of radiation — to unambiguously identify the *optical axis* with symmetry axis and make use of simplifications gained from symmetry. From the mathematical insight [9], we expect the caustic to be shaped as a revolution of cusp type catastrophe. Also, only spherical aberrations of (generally) all orders are expected to rise for any axisymmetric source.

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As a general relativistic specificity, the Legendre transformation, carrying eikonals of momentum and coordinate representations between each other, usually cannot be implemented analytically in a curved spacetime, due to mathematical complications. Thus it is of particular interest to be able to follow procedures, that utilise single representation formulae, or, to find connective formulae in Legendre transformation's stand, which is also the goal of this article.

Eikonal, rays, and caustic

Let there be a static spherically symmetric solution of Einstein equations, valid in spacetime region Σ. In spherical coordinates $(r,\ \vartheta,\ \varphi)$, this generally admits the metric

$$\Sigma:\ \mathrm{d}s^2 = g_{tt}(r)\, c^2 \mathrm{d}t^2 - g_{rr}(r)\, \mathrm{d}r^2 - r^2 (\, \mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\varphi^2)\,, \tag{1}$$

with c the speed of light. On a non-empty intersection σ with equatorial surface $\vartheta = \pi/2$ this brings

$$\Sigma|_{\vartheta=\pi/2} = \sigma:\ \mathrm{d}s^2 = g_{tt}\, c^2 \mathrm{d}t^2 - g_{rr}\, \mathrm{d}r^2 - r^2\, \mathrm{d}\varphi^2\,.$$

In the rest of the paper we shall restrict ourselves to this cross-section. Utilising the usual variable separation method [10], the solutions to the eikonal equation $\psi^{,j}\psi_{,j} = 0$ of the form (j, m being integers)

$$\Theta_\sigma : \psi - \psi_0 = \frac{\omega}{c} t - (-1)^k p_\varphi \varphi - (-1)^m \int \sqrt{\frac{\omega^2}{c^2}\frac{g_{rr}}{g_{tt}} - p_\varphi^2 \frac{g_{rr}}{r^2}}\, \mathrm{d}r \tag{2}$$

are its complete integrals on this cross-section, while their Hessian is non-zero. Consequently, (2) may serve as a (*momentum*) *eikonal*. Individual terms in previous equation change sign, whenever change in the direction of appropriate coordinate takes place along the path studied. Denote $a(p_\varphi)$ the root(s) of the last integrand. With the positive definiteness of metric coefficient(s) — as introduced in (1) — this yields

$$a(p_\varphi) : \frac{\omega^2}{c^2 g_{tt}(a)} - \frac{p_\varphi^2}{a^2} = 0\ . \tag{3}$$

The particular eikonal realising a testing field point source at $[r_\mathrm{s},\ \varphi_\mathrm{s}]$ is then

$$\begin{aligned} &\theta_\sigma : \psi(t,\, r,\, \varphi) - \psi_0(t_0,\, r_\mathrm{s},\, \varphi_\mathrm{s}) \\ &= \frac{\omega}{c}\,(t - t_0) - (-1)^k p_\varphi\, (\varphi - \varphi_\mathrm{s}) - (-1)^m \int_{r_\mathrm{s}}^{a} \mp \int_{a}^{r} \sqrt{g_{rr}(y) \left(\frac{\omega^2}{c^2 g_{tt}(y)} - \frac{p_\varphi^2}{y^2} \right)}\, \mathrm{d}y; \end{aligned} \tag{4}$$

the sign minus or plus between the two integrals depends on whether $r = a$ on the ray lies between the considered end-points of the ray, or not, respectively. The previous expression is also valid for regular points a, so (4) is valid for all points within considered segment of path. The notation itself is to be understood as

$$-\int_a^b - \int_c^d f(y)\,\mathrm{d}y \equiv -\left(\int_a^b f(y)\,\mathrm{d}y - \int_c^d f(y)\,\mathrm{d}y\right) .$$

Following now the canonical procedure, the particular *ray equation* is then $\pi_\sigma = \partial\theta_\sigma/\partial \mathrm{p}_\varphi$; upon using the Leibniz rule it reads

$$\pi_\sigma : \varphi - \varphi_\mathrm{s} = (-1)^{k+m} \int\limits_{r_\mathrm{s}}^{a} \mp \int\limits_{a}^{r} \frac{p_\varphi \sqrt{g_{rr}}}{y^2 \sqrt{\dfrac{\omega^2}{c^2 g_{tt}} - \dfrac{p_\varphi^2}{y^2}}}\,\mathrm{d}y$$
$$- (-1)^{k+m} \left(\frac{\partial a}{\partial p_\varphi} \pm \frac{\partial a}{\partial p_\varphi}\right) \left[\sqrt{g_{rr}\left(\frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{y^2}\right)}\,\right]_{y=r_0} . \qquad (5)$$

Due to the equation (3) for rootial points and regular end-points independence of p_φ, equation (5) generally simplifies to

$$\pi_\sigma : \varphi - \varphi_\mathrm{s} = (-1)^{k+m} \int\limits_{r_\mathrm{s}}^{a} \mp \int\limits_{a}^{r} \frac{p_\varphi \sqrt{g_{rr}}}{y^2 \sqrt{\dfrac{\omega^2}{c^2 g_{tt}} - \dfrac{p_\varphi^2}{y^2}}}\,\mathrm{d}y . \qquad (6)$$

The vanishing of last term in (5) has a simple interpretation: despite of acquired discontinuity of integrand at rootial points, there is no discontinuity of ray itself in any end-point. It also follows from (6) that

$$\left.\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right|_{r=a} = 0,$$

hence as long as a is a root of odd multiplicity, $r = a$ gets clear meaning of turning point on a ray: here, the ray radial coordinate difference must change sign to keep the square-rooted term non-negative for the ray to continue past this point.

The Legendre transformation to obtain the coordinate eikonal would mean to eliminate p_φ from (4) using (6). That, unfortunately, is generally not analytically possible. On the other hand, it is simple to eliminate $\varphi - \varphi_\mathrm{s}$, after which one obtains an *eikonal along ray* — the object closest to wavefront(s) description, that is generally available:

$$\lambda_\sigma : \psi_{p_\varphi} - \psi_0 = \frac{\omega}{c}(t - t_0) - (-1)^m \int\limits_{r_\mathrm{s}}^{a} \mp \int\limits_{a}^{r} \frac{\omega^2 \sqrt{g_{rr}}}{c^2 g_{tt} \sqrt{\dfrac{\omega^2}{c^2 g_{tt}} - \dfrac{p_\varphi^2}{y^2}}}\,\mathrm{d}y . \qquad (7)$$

Obtaining the *caustic* $\kappa_\sigma = \partial\pi_\sigma/\partial p_\varphi$ is not as straightforward as that of ray equation (5), for now there remains a dependence of integral rootial end-points on a derivation parameter, but discontinuity of the integrand in these end-points is added. This precludes the use of Leibniz rule — the way how to proceed with general calculation is to remove the parameter dependence of the end-point(s). This can be done separately in the two integrals of (6) by transformations

$$\xi_1 = \frac{y - r_{\mathrm{s}}}{a - r_{\mathrm{s}}}\,, \quad \xi_2 = \frac{y - a}{r - a}\,.$$

Differentiating the ray equation after transformations and consequently returning to the original variables, we finally obtain (with prime denoting differentiation with respect to the radial coordinate)

$$\begin{aligned}
&\kappa_\sigma : 0 = \\
&\int_{r_{\mathrm{s}}}^{a} \Big\{ g_{tt}(\omega^2 y^2 - c^2 g_{tt} p_\varphi^2) \left[2g_{rr}^2 \left(\frac{\partial a}{\partial p_\varphi} p_\varphi - a + r_{\mathrm{s}} \right) + \frac{\partial a}{\partial p_\varphi} p_\varphi (y - r_{\mathrm{s}})(g'_{rr} - 4g_{rr}^2) \right] \\
&+ p_\varphi g_{rr}^2 \left[2c^2 g_{tt}^2 p_\varphi y (a - r_{\mathrm{s}}) + \frac{\partial a}{\partial p_\varphi} (y - r_{\mathrm{s}})(\omega^2 g'_{tt} y^3 + 2c^2 g_{tt}^2 p_\varphi^2) \right] \Big\} \\
&\times \frac{\mathrm{d}y}{2(a - r_{\mathrm{s}}) g_{rr}^{3/2} g_{tt}^2 y^5 \left(\dfrac{\omega^2}{c^2 g_{tt}} - \dfrac{p_\varphi^2}{y^2} \right)^{3/2}} \\
&\mp \int_{a}^{r} \Big\{ g_{tt}(\omega^2 y^2 - c^2 g_{tt} p_\varphi^2) \left[2g_{rr}^2 \left(\frac{\partial a}{\partial p_\varphi} p_\varphi - a + r \right) + \frac{\partial a}{\partial p_\varphi} p_\varphi (y - r)(g'_{rr} - 4g_{rr}^2) \right] \\
&+ p_\varphi g_{rr}^2 \left[2c^2 g_{tt}^2 p_\varphi y (a - r) + \frac{\partial a}{\partial p_\varphi} (y - r)(\omega^2 g'_{tt} y^3 + 2c^2 g_{tt}^2 p_\varphi^2) \right] \Big\} \\
&\times \frac{\mathrm{d}y}{2(a - r) g_{rr}^{3/2} g_{tt}^2 y^5 \left(\dfrac{\omega^2}{c^2 g_{tt}} - \dfrac{p_\varphi^2}{y^2} \right)^{3/2}}\,.
\end{aligned} \tag{8}$$

Equation (8) is the sought one for caustic, if only last integrals uniformly converge. The transformations used were linear; other approaches are possible, *e.g.*, transformations of the type $y = a \pm \xi^2$ would remove the singularity of integrands in turning end-points. Of course, when there are no turning points present within the ray segment under consideration, the caustic from (2) is simply

$$0 = \int \frac{\omega^2 \sqrt{g_{rr}}}{c^2 g_{tt}} \frac{\mathrm{d}r}{r^2 \left(\dfrac{\omega^2}{c^2 g_{tt}} - \dfrac{p_\varphi^2}{r^2} \right)^{3/2}}\,. \tag{9}$$

As an example consider Minkowski spacetime, which can be covered by a single metric

$$\boldsymbol{R}^2 \times \boldsymbol{S}^2 = \Sigma :\ \mathrm{d}s^2 = c^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\, \mathrm{d}\vartheta^2 - r^2 \sin^2\vartheta\, \mathrm{d}\varphi^2\,.$$

According to (6), the ray equation on the equatorial section is in such a case

$$\pi_\sigma : \varphi - \varphi_{\rm s} = (-1)^{k+m} \int_{r_{\rm s}}^{a} \mp \int_{a}^{r} \frac{p_\varphi}{y^2} \frac{\mathrm{d}y}{\sqrt{\dfrac{\omega^2}{c^2} - \dfrac{p_\varphi^2}{y^2}}} \, .$$

To be able to profit from the general construction developed, we shall not explicitly calculate the last integral. In that way, at last, the caustic as (8) becomes

$$\kappa_\sigma : (-1)^{k+m} \left(\frac{1}{\sqrt{r^2 - \varrho^2}} \pm \frac{1}{\sqrt{{r_{\rm s}}^2 - \varrho^2}} \right) = 0,$$

with $\varrho = (p_\varphi c)/\omega$ non-negative without loss of generality. It is now clearly seen, that before the ray turning point of $r = \varrho$ as from (3), there lies the only caustic point — the source itself at $r = r_{\rm s}$ (the second coordinate $\varphi = \varphi_{\rm s}$ is obtained from the ray equation stated above in this example). After turning point, there are no caustic points at all (in correspondence with the beams constant divergence). Also, for the calculation presented, $r \geq \varrho$ has to hold. In this way, in the flat case, ϱ has directly the meaning of the ray closest advance point towards origin (i.e. turning point) radial coordinate. In the further, we stick to this notation and shall label the rays by ϱ.

The geometry

Let the solution of interest of Einstein equations consist of two metrics, properly sewed on $r = r_0$, with the point source of radiation at $[r_{\rm s},\ \varphi_{\rm s}]$ lying not in the inner region $r_{\rm s} \geq r_0$.

The particular eikonal for a general ray passing into the inner region at $[r_0,\ \varphi_{\rm in}]$ and leaving it subsequently at $[r_0,\ \varphi_{\rm out}]$ after passing the turning point of $r = a$, as visualised by Fig. 1, becomes

$$\begin{aligned} \theta : \psi - \psi_0 \, = \, & \frac{\omega}{c}(t - t_0) - (-1)^k \frac{\omega}{c} \varrho(\varphi - \varphi_{\rm s}) \\ & - (-1)^m \frac{\omega}{c} \int_{r_{\rm s}}^{r_0} - \int_{r_0}^{r} \Theta_{\rm outer}(y)\, \mathrm{d}y - 2(-1)^m \frac{\omega}{c} \int_{r_0}^{a} \Theta_{\rm inner}(y)\, \mathrm{d}y \, , \end{aligned}$$

where

$$\Theta(y) \equiv \sqrt{g_{rr}(y) \left(\frac{1}{g_{tt}(y)} - \frac{\varrho^2}{y^2} \right)}$$

are integrands from (4), with the subscript choosing solution, whose metric coefficients are appropriate. Then the particular ray equation reads

$$\pi : \varphi - \varphi_{\rm s} = -(-1)^{k+m} \int_{r_{\rm s}}^{r_0} - \int_{r_0}^{r} \Pi_{\rm outer}(y)\, \mathrm{d}y - 2(-1)^{k+m} \int_{r_0}^{a} \Pi_{\rm inner}(y), \qquad (10)$$

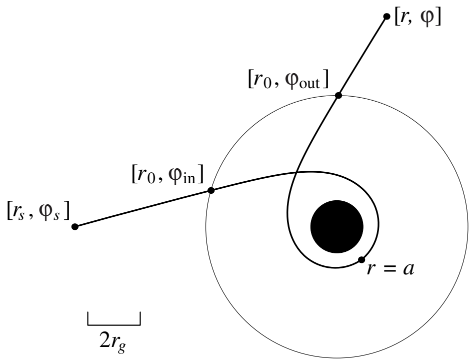

Fig. 1. The sketch of the geometrical situation concerning a ray (thick curve): within the great circle the inner solution is valid, with the black ring showing extent of mass-critical radius. Note, that no further scaling information is needed, if radial coordinate is expressed in critical radii.

where

$$\Pi(y) = \frac{\partial \Theta(y)}{\partial \varrho} \equiv \frac{\varrho\sqrt{g_{rr}(y)}}{y^2\sqrt{\dfrac{1}{g_{tt}(y)} - \dfrac{\varrho^2}{y^2}}}$$

are integrands from (6). In this way, only those situations, when rays, that enter the inner solution region, exhibit in it its (single) turning point and after leaving to outer one, they (from symmetry) show no other turning points, are taken into account.

Let us now formally evaluate the caustic $\kappa = \partial\pi/\partial\varrho$. As there are no turning points within outer solution, the integrands exhibit no singularities up to its border as well as the integration end-points are simply constant there. Thus,

$$\kappa : \int_{r_\mathrm{s}}^{r_0} - \int_{r_0}^{r} K_{\mathrm{outer}}(y)\,\mathrm{d}y + 2\frac{\partial}{\partial \varrho}\int_{r_0}^{a} \Pi_{\mathrm{inner}}(y)\,\mathrm{d}y = 0,$$

where

$$K(y) = \frac{\partial \Pi(y)}{\partial \varrho} \equiv \frac{\sqrt{g_{rr}(y)}}{g_{tt}(y)} \frac{1}{y^2\left(\dfrac{1}{g_{tt}(y)} - \dfrac{\varrho^2}{y^2}\right)^{3/2}}$$

are (up to a constant) integrands as in (9). The lengthy calculation according to (8) is not required in second term of caustic, if we happen to know the value J of the full angular accrument along the ray within the inner solution analytically:

$$2\int_{r_0}^{a} \Pi_{\mathrm{inner}}(y)\,\mathrm{d}y = J(\varrho);$$

in that case, we finally obtain the equation of caustic in the form

$$\varphi-\varphi_{\mathrm{s}} = -(-1)^{k+m}J-(-1)^{k+m}\int\limits_{r_{\mathrm{s}}}^{r_0}-\!\!\int\limits_{r_0}^{r}\frac{\varrho\sqrt{g_{rr}}}{y^2\sqrt{\dfrac{1}{g_{tt}}-\dfrac{\varrho^2}{y^2}}}\,\mathrm{d}y$$

$$0 = \frac{\partial J}{\partial\varrho}+\int\limits_{r_{\mathrm{s}}}^{r_0}-\!\!\int\limits_{r_0}^{r}\frac{\sqrt{g_{rr}}}{g_{tt}}\frac{\mathrm{d}y}{y^2\left(\dfrac{1}{g_{tt}}-\dfrac{\varrho^2}{y^2}\right)^{3/2}},$$

where all metric coefficients present belong to the outer solution. To obtain caustic in parametric form, the second of equations must be understood as implicit equation for $r(\varrho)$, and subsequently the first one as an explicit equation for $\varphi(\varrho, r(\varrho))$. Even though ϱ is not generally the value of turning point radial coordinate, still, $\varrho=0$ is the only ray passing through origin. Thus, an expansion in the vicinity of optical axis (coming from symmetry) is acquired by expanding the coordinates for small ϱ. Using implicit derivatives formulae we obtain general (parametric) expression of caustic

$$\begin{aligned}
r(\varrho) &= r(0)+\frac{r(0)^2}{g_{tt}\sqrt{g_{rr}}}\frac{\partial^2 J}{\partial\varrho^2}\bigg|_0\varrho+\frac{r^3(0)}{g_{tt}^3 g_{rr}}\left[\left(\frac{\partial^3 J}{\partial\varrho^3}\bigg|_0+3\int\limits_{r_{\mathrm{s}}}^{r_0}-\!\!\int\limits_{r_0}^{r(0)}\frac{\sqrt{g_{rr}}g_{tt}^2}{y^4}\,\mathrm{d}y\right)\frac{\sqrt{g_{rr}}g_{tt}^2}{r(0)}\right.\\
&\quad\left.-\frac{g_{tt}g_{rr}'r(0)+2r(0)g_{rr}g_{tt}'-4g_{rr}g_{tt}}{2g_{rr}}\left(\frac{\partial^2 J}{\partial\varrho^2}\bigg|_0\right)^2\right]\frac{\varrho^2}{2}+\dots,\\
\varphi(\varrho) &= \left(\varphi_{\mathrm{s}}-(-1)^{k+m}J(0)\right)-(-1)^{k+m}\left(\frac{\partial J}{\partial\varrho}\bigg|_0+\int\limits_{r_{\mathrm{s}}}^{r_0}-\!\!\int\limits_{r_0}^{r(0)}\frac{\sqrt{g_{tt}g_{rr}}}{y^2}\,\mathrm{d}y\right)\varrho\\
&\quad-(-1)^{k+m}\left(1+\frac{2}{\sqrt{g_{tt}}}\right)\frac{\partial^2 J}{\partial\varrho^2}\bigg|_0\frac{\varrho^2}{2}+\dots
\end{aligned}\tag{11}$$

with $r(0)\equiv r|_{\varrho=0}$ defined implicitly as

$$r(0):\int\limits_{r_{\mathrm{s}}}^{r_0}-\!\!\int\limits_{r_0}^{r}\frac{g_{tt}\sqrt{g_{rr}}}{y^2}\,\mathrm{d}y+\frac{\partial J}{\partial\varrho}\bigg|_0=0\,;$$

outside integrals, all metric coefficients in last equations are to be treated as evaluated in $r(0)$.

The equation of projection of eikonal along ray (7) is

$$\lambda : \psi_\varrho - \psi_0 = -(-1)^m \frac{\omega}{c} \int_{r_\mathrm{s}}^{r_0} - \int_{r_0}^{r} \Lambda_\mathrm{outer}(y)\,\mathrm{d}y - 2(-1)^m \frac{\omega}{c} \int_{r_0}^{a} \Lambda_\mathrm{inner}(y)\,\mathrm{d}y,$$

where

$$\Lambda(y) = \Theta(y) + \varrho \frac{\partial \Theta(y)}{\partial \varrho} \equiv \frac{\sqrt{g_{rr}}}{g_{tt}\sqrt{\dfrac{1}{g_{tt}} - \dfrac{\varrho^2}{y^2}}}$$

are (up to a constant) integrands from (7). Taking now the equation $\psi_\varrho = \mathrm{const}$ of constant phase accrument along ray for implicit expression for $r(\varrho)$ of the wavefront (into the constant, the signs and factor ω/c are set to stick to geometrical substantiality of wavefront) and using ray equation similarly to the case of caustic, one finally obtains the parametric expression of wavefront in the form

$$\begin{aligned}
r(\varrho) &= r(0) + \sqrt{\frac{g_{tt}}{g_{rr}}} \frac{\partial I}{\partial \varrho}\bigg|_0 \varrho \\
&\quad + \sqrt{\frac{g_{tt}}{g_{rr}}} \left[\frac{\partial^2 I}{\partial \varrho^2}\bigg|_0 + \int_{r_\mathrm{s}}^{r_0} - \int_{r_0}^{r(0)} \frac{\sqrt{g_{rr} g_{tt}}}{y^2}\,\mathrm{d}y - \frac{1}{2}\left(\frac{\partial I}{\partial \varrho}\bigg|_0\right)^2 \frac{g'_{rr} g_{tt} - g_{rr} g'_{tt}}{\sqrt{g_{rr}}^3 \sqrt{g_{tt}}} \right] \frac{\varrho^2}{2} + \ldots, \\
\varphi(\varrho) &= \Big(\varphi_\mathrm{s} - (-1)^{k+m} J(0)\Big) - (-1)^{k+m} \left(\frac{\partial J}{\partial \varrho}\bigg|_0 + \int_{r_\mathrm{s}}^{r_0} - \int_{r_0}^{r(0)} \frac{\sqrt{g_{tt} g_{rr}}}{y^3}\,\mathrm{d}y \right) \varrho \\
&\quad - (-1)^{k+m} \left(1 - 2 \frac{g_{tt}}{r(0)^2} \frac{\partial I}{\partial \varrho}\bigg|_0 \right) \frac{\partial^2 J}{\partial \varrho^2}\bigg|_0 \frac{\varrho^2}{2} + \ldots
\end{aligned} \tag{12}$$

with $r(0)$ implicitly defined now from

$$-\int_{r_\mathrm{s}}^{r_0} - \int_{r_0}^{r(0)} \sqrt{\frac{g_{rr}}{g_{tt}}}\,\mathrm{d}y - I(0) = \mathrm{const.}\,;$$

$I(\varrho)$ is the ray total phase accrument along ray within the inner part of solution,

$$2 \int_{r_0}^{a} \Lambda_\mathrm{inner}(y)\,\mathrm{d}y = I(\varrho)\ .$$

Note the connection between the two parametric expressions: the one for caustic (11) and that of wavefront (12). This connection, as suggested by the fact that both make use the ray equation (10), reduces in the flat case to the fact that the caustic is an *evolute* of the wavefronts. For more detailed discussion, see [11].

Particular metric and the results

Let us choose for particular configuration the inner constant mass-density uncharged fluid solution and the outer, Schwarzschild one [12], with the field of electromagnetic point source testing. On the equatorial section $\vartheta = \pi/2$ we obtain

$$\mathrm{d}s^2 = \left[\frac{3}{2}\sqrt{1-\frac{r_\mathrm{g}}{r_0}} - \frac{1}{2}\sqrt{1-\frac{r_\mathrm{g}r^2}{{r_0}^3}}\right]^2 c^2\mathrm{d}t^2 - \frac{1}{1-\dfrac{r_\mathrm{g}r^2}{{r_0}^3}}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\varphi^2, \qquad r \le r_0$$

$$\mathrm{d}s^2 = \left(1-\frac{r_\mathrm{g}}{r}\right)c^2\mathrm{d}t^2 - \frac{1}{1-\dfrac{r_\mathrm{g}}{r}}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\varphi^2, \qquad r \ge r_0\,.$$

with r_g the critical radius of matter involved, $r_\mathrm{g} = (2MG/c^2)$ in SI units. This choice can serve as a reasonable depiction of an astrophysical configuration, moreover, the (dimension-independent) perfect fluid metric allows for analytical results.

Within the Schwarzschild solution, all integrals involved in (4)–(8) are elliptic, approving the introduction. However, owing to the advantage of focus definition, the reduced integrals within expansion coefficients of (11), (12) are elementary. Recalling that $\varrho \ge 0$ was chosen, we state the turning points for metrics chosen,

$$a_\mathrm{Schw} = \frac{2\varrho}{\sqrt{3}}\cos\left(\frac{\pi}{3} - \frac{1}{3}\arccos\frac{3\sqrt{3}}{2\varrho}\right),$$

$$a_\mathrm{fl} = \frac{3\dfrac{r_0}{\varrho}\sqrt{1-\dfrac{r_\mathrm{g}}{r_0}} - \sqrt{\dfrac{{r_0}^2}{\varrho^2} - \dfrac{2r_\mathrm{g}}{r_0} + \dfrac{9r_\mathrm{g}^2}{4{r_0}^2}}}{2\dfrac{r_0}{\varrho^2} + \dfrac{r_\mathrm{g}}{2{r_0}^2}}\,.$$

It is a matter of lengthy discussion of technical kind, that their behaviour is as expected in previous section, *i.e.* for rays closing to origin from high radial values, the stated expressions are the only turning points present; moreover, for all rays that are to enter fluid, $a_\mathrm{Schw} < r_0$ holds. More interestingly, for a fluid chosen,

$$J = -\pi - \sum_{\pm}\arcsin\frac{\dfrac{3\varrho^2 r_\mathrm{g}(r_0-r_\mathrm{g})}{\sqrt{r_0}} \mp \varrho^2 r_\mathrm{g}\sqrt{r_0-r_\mathrm{g}} - 2{r_0}^3\left(\sqrt{r_0}\mp\sqrt{r_0-r_\mathrm{g}}\right)}{r_0\left(\sqrt{r_0-r_\mathrm{g}}\mp\sqrt{r_0}\right)\sqrt{4{r_0}^4 - 8\varrho^2 r_\mathrm{g}r_0 + 9\varrho^2 r_\mathrm{g}^2}},$$

where the double signs stand for summing two terms within J, once with upper signs and once with the lower ones. Then $r|_{\varrho=0}$ for caustic becomes

$$r(0) = \frac{{r_0}^2 r_\mathrm{s}}{3r_\mathrm{g}r_\mathrm{s} - {r_0}^2}, \tag{13}$$

which is the value of the radial coordinate of the caustic axial point, *i.e.* the radial position of the lens focus. Note that though emerged from expansions, by definition of the focus, this value is exact. Following the calculations, we can write for the metrics chosen

$$I = -4\frac{\omega {r_0}^2}{c r_{\rm g}\sqrt{8\dfrac{r_0}{r_{\rm g}} - 9}}\left(\frac{\pi}{2} + \arcsin\frac{r_0^{3/2}(3r_{\rm g} - 2r_0)}{\sqrt{r_0 - r_{\rm g}}\sqrt{4{r_0}^4 + 9\varrho^2 r_{\rm g}^2 - 8\varrho^2 r_{\rm g} r_0}}\right) .$$

Adopting another result from [11], the general form of axial wavefront $h = \mathrm{const}$ within Schwarzschild geometry is

$$\begin{aligned} h &= [r + r_{\rm g}\ln(r - r_{\rm g}) + c_0] \\ &\quad + \frac{r}{-2 + rc_2}(\varphi - \varphi_A)^2 + \frac{-\frac{1}{2}r_{\rm g} + \frac{2}{3}r + r^4 c_4}{(-2 + rc_2)^4}(\varphi - \varphi_A)^4 + \dots , \end{aligned} \tag{14}$$

i.e., to completely describe such wavefront, a single constant in every order of expansion is to be specified. To find the value of φ_A around which to expand the wavefront, we proceed as follows. The choice of point source has unambiguously given rise to optical axis as coordinate line passing through source and origin. The optical axis is thus realized by ray $\varrho = 0$ which gives *e.g.* from (12)

$$\varphi = \varphi_s - (-1)^{k+m} J(0) = \varphi_s + (-1)^{k+m}\pi ,$$

which, indeed, is the continuation of coordinate line $\varphi = \varphi_{\rm s}$. Note, that the same holds for caustic (11), *i.e.* the cusp of a caustic, which is also the focus point, lies on this axis, as anticipated in introduction. The integers k, m also confirm to rule the orientation of the ray(s).

Substituting into general expression (14) the equation (12) for eikonal along ray and re-expanding in powers of ϱ, enables us to find the values of aberration constants in the form

$$\begin{aligned} c_0 &= -I(0) - 2[r_0 + r_{\rm g}\ln(r_0 - r_{\rm g})] + r_{\rm s} + r_{\rm g}\ln(r_{\rm s} - r_{\rm g}) , \\ c_2 &= 2\frac{3r_{\rm s}r_{\rm g} - {r_0}^2}{{r_0}^2 r_{\rm s}} , \\ c_4 &= -\frac{1}{3{r_0}^6}(81r_{\rm g}^3 - 108r_{\rm g}^2 r_0 + 15r_{\rm g}{r_0}^2 + 16{r_0}^3) + \frac{2}{3{r_{\rm s}}^3} - \frac{1}{2}\frac{r_{\rm g}}{{r_{\rm s}}^4} , \\ &\vdots \end{aligned} \tag{15}$$

To find the wave aberration in terms of wave-progress difference, let us state the aberration coefficients within Schwarzschild solution for axial point source $[r_s', \varphi_A]$ wavefronts $h' = \mathrm{const}'$ before turning point:

$$\begin{aligned} c_0' &= \mathrm{const}' + r_s' + r_{\rm g}\ln(r_s' - r_{\rm g}) , \\ c_2' &= \frac{2}{r_s'} , \\ c_4' &= \frac{1}{6}\frac{3r_{\rm g} - 4r_s'}{{r_s'}^4} , \\ &\vdots \end{aligned}$$

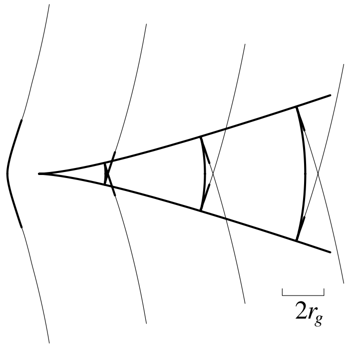

Fig. 2. The situation for far-source wavefronts near detail of caustic (thick curve). Depicted are the phase equi-spaced wavefronts, the bold segments showing the inner region traversed part. The ϱ parameter extent is the same for all wavefronts shown. As can be noted, the caustic indeed serves as a set of wavefronts singularities. Also note that the orientation of caustic is opposite to the case of reflection on a spherical mirror. In other words, the spherical aberration (of lowest order at least) for the cluster is of opposite sign to the mirror one.

Now, as a basis of aberration formulation, the first two terms in expansion of wave-progress difference

$$\begin{aligned} h - h' &= (c_0 - c_0') + \frac{r^2(c_2' - c_2)}{(-2 + rc_2)(-2 + rc_2')}(\varphi - \varphi_A)^2 \\ &+ \left[\frac{-\dfrac{r_\mathrm{g}}{2} + \dfrac{2r}{3} + r^4 c_4}{(-2 + rc_2)^4} - \frac{-\dfrac{r_\mathrm{g}}{2} + \dfrac{2r}{3} + r^4 c_4'}{(-2 + rc_2')^4} \right] (\varphi - \varphi_A)^4 + \ldots \end{aligned} \tag{16}$$

can be always annihilated by suitable choice of reference (point) source position and phase. Particularly here, for any r_s', the const$'$ can be set to equal $c_0 = c_0'$, and, confronting the second aberration coefficients, we obtain

$$c_2 = c_2' : \qquad r_s' = \frac{{r_0}^2 r_\mathrm{s}}{3r_\mathrm{s} r_\mathrm{g} - {r_0}^2}$$

in direct agreement with (13). In this way, within Gaussian optics, the focus point is the apparent point source for emerging wavefronts (see Fig. 2). The first non-zero term gives rise to the wave-progress difference expansion

$$h - h' = \frac{r^4(c_4 - c_4')}{(-2 + rc_2)^4}(\varphi - \varphi_A)^4 + \ldots \tag{17}$$

which is the lowest spherical aberration term. The behaviour of higher order terms is similar: the condition that the wave-progress difference is zero only, if the wavefronts are identical ($c_k = c_k'$) is manifest; note, however, that for such behaviour, the general possibility of identification of the two lowest aberration coefficients is crucial. This behaviour forms the fundaments of Gaussian optics.

Conclusion

In this paper, the author presented a general way of mathematically rigorous manipulation with optical ideas of focusing via caustic study within the frame of the general relativity. Expressions for caustic (11) and wavefront in the sense of the eikonal along ray (12) for testing electro-magnetic field on the equatorial section of static spherically symmetric space-time were obtained. As a consequence, upon choosing particular configuration, the exact value of perfect fluid lens focus (13) was given for a testing-field point source,

$$r_f \approx \frac{{r_0}^2}{3r_{\mathrm{g}}},$$

here in the far-source limit (see Fig. 3).

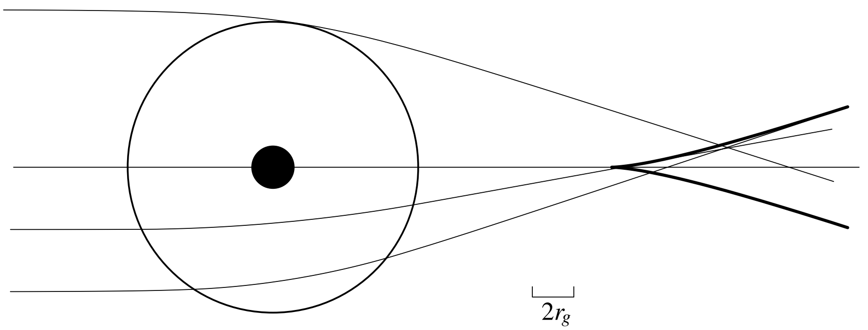

Fig. 3. The caustic (thick curve) situation of cluster with $r_0 = 7r_{\mathrm{g}}$ for far-source configuration. The caustic cusp point (the focus of a system) is then at $r = 16^1\!/_3 r_{\mathrm{g}}$. Several inner rays are shown, all inevitably touching the caustic, that actually extends to radial infinity before touching the boundary ray. Hence, despite the diffractional corrections, the optical influence of cluster intervenes in a significant range of ambient universe.

Also, the constants (16) in the aberration expansion of the wavefront were obtained, moreover, solely using momentum representation formulae. In addition, a comparison with point-source aberrations was performed, confirming the Gaussian position of focus. The expansion of wave-progress difference (17) was acquired, which starts with the lowest spherical aberration term, as expected.

Making use of the general treatment presented in the paper, results for different configurations are easily obtainable.

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