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# Electromagnetic Waves in a Gravitational Field

PhD thesis

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*PER ASPERA, AD ASTRA !*

I would like to thank the supervisor of this work, prof. Michal Lenc, for doing all those things, that allowed for this work to exist.

# Preface

**Gravitational Lensing**
Although the affection of light by gravitational field was proposed rather long ago by scientists (e.g. Laplace, [**Lap**]), mostly in analogy to massive particles behaviour, the actual *gravitational lensing* has been born with Einstein's work [**Ein**] from 1915 regardless of the fact, that the designation itself was incorporated by Lodge, 1919; the measurement of Hyades stars apparent position in geometrical vicinity of Sun during 1919 eclipse is famous.

The gravitational lensing, as comprehensively described e.g. in [**Sch**] is very successful in providing reasonable results. There are two main regimes in which lensing occurs: the *strong* one [**Ehl**], with multiple images rising and the *weak* one [**Bar**], when the light is bent to but merely distort the beams. The first regime allows mainly to measure local properties of deflectors [**Koc**] and provides the valuable information on statistics within their populations. One example for all let be the Einstein cross 2237+0305 at $z \approx 1.695$ [**Huc**]. The second one probes the large scale distribution of space inhomogeneities in a very effective way [**Wae**] by measuring the share spectrum. In addition, *microlensing* provides us with possibility of seeking for extra-solar planets [**Alc**]; the projects MACHO and OGLE must be stated here.

Recently, the trend of mathematically more sophisticated treatment appears, see e.g. [**Pet**], mostly using the caustic approximative of sources extended surface to observer sky map[**Mol**]. Though very potent, most of these approaches are however themselves only further approximations to relativistic optics – which is hereby defined as extension of geometrical optics [**Bor**] to curved case, i.e. the covariant eikonal equation and lowest-order amplitude transfer covariant equation. In this work, parts of aberrational formulation to relativistic optics (which is exact from viewpoint of gravitational lensing) are studied.

**Geometrical Optics**
During the several hundred years of optics evolution, the aberration formulation has been established as the most suitable and versatile mean of optical systems properties depiction. After Petzval, the first systematical treatment of aberrational structure of wavefront was given by Seidel, 1856 [**Sei**]. Nowadays, this treatment is utilised in the whole range from public optical cameras production to charged particle optical instruments construction.

Many principal ideas of classical optics are valid in curved spacetimes as well, however there are some, that cannot be treated therein. Yet, some of the latter can be redefined in a generalised way, such that they were meaningful in curved cases and reducing to well established ones in the flat case. The idea of (positive) focusing is abstract enough in this sense, when requiring touching (in mathematical sense) of adjacent rays to occur. As the focusing is of prime interest to all optics, we show in this work, how this can be dealt with via caustic study within curved spacetimes.

## Notation and Conventions

*index sized mathematics in regular positions:*

$f(y-a)$ function arguments, where reasonable to avoid misunderstanding; also, to distinguish from $f\cdot(y-a)$

*roman versus bold mathematical (in)equations:*

$r=a$ particular value of e.g. radial coordinate

$\boldsymbol{\vartheta=\pi/2}$ geometrical object, in this case, equatorial (hyper)surface

$\mathrm{d}f=0$ triviality of a form, i.e. $f_{,i}=0$

$\mathbf{d}\boldsymbol{f}\mathbf{=0}$ codimension one geometrical object, here a differential equation of a curve

*'addition' of integrals*

$$-\int_a^b \pm \int_c^d f(y)\mathrm{d}y \quad = -\left(\int_a^b f(y)\mathrm{d}y \pm \int_c^d f(y)\mathrm{d}y\right)$$

*derivatives*

$g_{,t} \equiv \dfrac{\partial g}{\partial x^t}$ partial derivative

$h^{;t} \equiv \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial x_t}(\sqrt{g}h)$ covariant derivative

$f_z \equiv \dfrac{\partial f}{\partial z}$ however, where misleading is not possible only

The (Einstein) summation notation is used in non-roman indices. In the work, geometrical units ($c=1$) are used, with the exception of utilising SI units where stated.

## Typography

The main body of the work is set in full width. It contains all the derivations and main results. **In bold face, the highlights are typed**.

> *In italic, examples, usually showing the validity of derived facts on basic simplifications are given. Unless stated otherwise, the flat case is considered with $x$, $y$, $z$ denoting the Cartesian coordinates as well as $x^i$; or ($r$, φ, ϑ) meaning the spherical coordinates with most commonly ($r$, φ) being the polar coordinates on the equatorial section $\boldsymbol{\vartheta=\pi/2}$.*

In other items, the usual typography conventions are obeyed.

# Introduction

This work deals with the procedure of establishing the geometrical optics extension into curved spacetimes, i.e. the *covariant optics*; in the course of the work an aberration formulation of covariant optics is developed.

The ideas are (re)constructed in such a way, that where the generalisations of well known ones are needed, the definitions are provided in such a way, that these new objects and/or definitions reduced to the well known forms from the flat case.

From the viewpoint of post-classical physics, optical quantities are divided into two groups: the ones, that are affected by post-classicity (e.g. index of refraction, as discussed below) and the ones that are left intact. The latter ones, among which the rays – that pass through fixed points of spacetimes irrespective of observers – belong, can be called *primary* quantities. From the same reason, (perfect) focusing belongs here to. The primary quantities are of main interest to this work.

Owing to the fact that the most concern of this work is the creating of mathematical constructions from basic parts of well-known physical theories, we state as little theory as possible; this allows to keep the structure of the work compact and to devote attention in greater detail to showing the consequences of these constructions and their connection with thoroughly known results on circumstances that from the viewpoint of this work appear as special cases.

It is also to be stated, that the work itself is a theoretical one, i.e. the most important are considered the calculations that can be hereby provided analytically. For this reason, the results are kept on strictly theoretical level, however, all the constructions and calculations presented are made ready for direct application to observational data in the case of interest.

To get a glance of the scope of problems touched by this work, the reader might want look first to the section of ***Exercise and Applications***, where the main results from ***Part One*** and ***Part Two*** are utilised.

## Maxwell Equations and Debye Procedure in a General Spacetime

However much useful the four-potential one-form A may be in electro-magnetic field definition, its use within general relativity must be careful. Also, while it appears preferable to solve within a spacetime of dimension $n$ a system of $2n$ PDEs of first order rather than one of $n$ PDEs of second order, we will make use of the Maxwell equations in form with the electro-magnetic field two-form F. The benefit resides also in its calibration invariance. If the four-potential description becomes desired, obviously F=dA with Lorentz calibration $A^{i}_{;i}$=0 is valid.

We restrict our attention to electro-vacuum *Maxwell equations*

$$* \,\mathrm{dF} = 0 \quad \wedge \quad * \,\mathrm{d} * \mathrm{F} = 0, \tag{1}$$

over metric manifold $(M,\, \mathrm{g})$, where $*$ stands for differential forms dualisation using complex Hodge star: to the general definition

$$* : \omega \in \Lambda^{p}_{\mathrm{C}} T^{*}M \rightarrow \eta \in \Lambda^{(m-p)}_{\mathrm{C}} T^{*}M \mid ** \equiv (-1)^{p(m-p)}\mathrm{id}$$

with $m$=dim$_{\mathrm{M}} \geq p$, the coordinate evaluation reads

$$\eta_{\imath_1..\imath_{m-p}} = g_{\imath_1 \ell_1} .. g_{\imath_{m-p} \ell_{m-p}} \, \varepsilon^{\ell_1 .. \ell_{m-p} \jmath_1 .. \jmath_p} \, \overline{\omega}_{\jmath_{1..p}} \, . \tag{2}$$

Although the version without complex conjugation is also valid, we stick to the expression stated above.

We point out the well-known fact, that the number of independent equations within (**1**) is only six, as well as the number of unknown is. Hence, it is sufficient to treat only three components from each set. Considering the structure of the equations, it appears a good idea to omit the both time components. Also note, that the system (**1**) is insensitive to dualisations of electro-magnetic field tensor F: a physical interpretation in cases it makes sense to speak of them, is that after interchanging the role of electric and magnetic parts between each other, the field as a whole remains to be a solution to (**1**).

The usual intensities describing the electro-magnetic fields can be for observer along a world-line with four-velocity u defined as components of one-forms E, H, where

$$\mathrm{F} = \mathrm{E} \wedge u + *(\mathrm{H} \wedge u).$$

For the intensities themselves we after little operation have

$$\mathrm{E} = *(u \wedge *\mathrm{F}) \qquad \mathrm{H} = *(u \wedge F)\ .$$

We shall solve the Maxwell equations (**1**) using the *Debye procedure*: we expect F in a form of expansion

$$F = \left[F^0 + \tfrac{\lambda}{i}F^1 + \left(\tfrac{\lambda}{i}\right)^2 F^2 + \cdots\right] \mathrm{Re}\, e^{i\psi/\lambda}$$

in powers of a small parameter $\lambda$, where $F^n \in \Lambda^2_C T^*M$ as we admit polarisation effects. Setting the last ansatz into Maxwell equations, we obtain

$$*dF = \left\{ \tfrac{i}{\lambda} *(d\psi \wedge F^0) + \sum_{n=0}^{\infty} \left(\tfrac{\lambda}{i}\right)^n \left[*dF^n + *(d\psi \wedge F^{n+1})\right] \right\} \mathrm{Re}\, e^{i\psi/\lambda} = 0$$

$$*d*F = \left\{ \tfrac{i}{\lambda} *(d\psi \wedge *F^0) + \sum_{n=0}^{\infty} \left(\tfrac{\lambda}{i}\right)^n \left[*d*F^n + *(d\psi \wedge *F^{n+1})\right] \right\} \mathrm{Re}\, e^{i\psi/\lambda} = 0$$

Regrouping the terms, we get for the terms of same power to parameter $\lambda$the system

$$\begin{array}{ll} \lambda^{-1} : & d\psi \wedge F^0 = 0,\ d\psi \wedge *F^0 = 0 \\ \lambda^{n \geq 0} : & dF^n + d\psi \wedge F^{n+1} = 0,\ d*F^n + d\psi \wedge *F^{n+1} = 0 \end{array}$$

where the dualisation was performed where necessary, without loss of generality. Indeed, $*u=0 \Leftrightarrow u=0$: 1) $u=0 \Rightarrow *u=0$ is evident from the definition; 2) Let $v=*u=0$. Then $*v=0$ according to 1), however, $*v=\pm u$ from definition. We shall henceforward use the dualisation order freely.

In components, the lowest order equations are

$$\psi_{,[\alpha} F^0_{\beta\gamma]} = 0 \qquad \psi'^{\kappa} F^0_{\tau\kappa} = 0,$$

respectively. From the independent subset from the first set we can find e.g. the axial components of $F$,

$$\psi_{,0} F^0_{ab} = F^0_{0[b} \psi_{,a]} \ .$$

Plugging this expression into the second set, we obtain (in matrix notation)

$$\left[\psi'^{\imath} \psi_{,\jmath} - (\psi'^{k} \psi_{,k}) \delta^{\imath}_{\jmath}\right] F^0_{0\imath} = 0 \ .$$

This homogeneous square system of linear equation will have non-trivial solution only if its determinant vanishes:

$$\psi'^{0} \psi_{,0} (\psi'^{k} \psi_{,k})^2 = 0$$

Hence, we can for the propagating waves ($\psi_{,0} \neq 0$) corollary, that the lowest order of Debye expansion implies the *eikonal equation*

$$\psi'^{k} \psi_{,k} = 0 \qquad (\mathbf{3})$$

and also the orthogonality of propagation direction and the wavefronts: electromagnetic waves are transversal.

As can be seen by differentiating (**3**), the rays also form the *isotropic geodesics* thanks to fact, that light congruencies are scalar ($k^a = \psi^{,a} \Rightarrow k^{a;b} = k^{b;a}$).

The formulas of the successive order of Debye expansion bring the lowest-order amplitude transport equation, e.g. in the form of *polarisation vector* parallel transport along the rays and the scalar amplitude transport. In this work, however, we will be mostly concerned by focusing of light and we will adopt another tool of studying the geometrical optics light intensities divergences.

Thus, these two lowest orders form a smallest consistent approximation of Maxwell equations, which is called a *geometrical optics*. It is however important to notice, that the higher corrections affect the amplitudes only, not the eikonal.

The formulas derived above can describe a testing electro-magnetic field within a general spacetime.

## The Particular Metrics Used

In this work, two main groups of metrics are used as ones of sufficient generality. First, it is the general $(n+1)$-dimensional stationary metric

$$\mathrm{d}s^2 = g_{\mathrm{tt}}(x^a)\,\mathrm{d}t^2 - 2g_{\mathrm{t}a}(x^a)\,\mathrm{d}t\,\mathrm{d}x^a - g_{ab}(x^a)\,\mathrm{d}x^a\,\mathrm{d}x^b, \qquad a,\ b = 1,\ \ldots,\ n \tag{4}$$

which comprises the most important insular solutions (Kerr-Newman black hole with all its special cases including (the flat) Minkowski spacetime). The most important special case $g_{\mathrm{t}a}=0$ of metric (**4**) is the general static spacetime

$$\mathrm{d}s^2 = g_{\mathrm{tt}}\,\mathrm{d}t^2 - g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b\ .$$

Secondly, in some parts of the work, it is the metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)(\,\mathrm{d}w^2 + f_k^2(w)(\,\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)), \tag{5}$$

where

$$f_k(w) = \begin{cases} \dfrac{1}{\sqrt{k}}\sin(\sqrt{k}w) & k>0,\ w\in\langle 0,\pi\rangle \\ w & k=0,\ w\in\langle 0,1\rangle \\ \dfrac{1}{\sqrt{-k}}\sinh(\sqrt{-k}w) & k<0,\ w\in\langle 0,\infty)\ . \end{cases}$$

which comprises the cosmological (FLRW) solutions. A common special case to both previous metrics is the flat spacetime.

Often, general static $d$-spherically symmetric sub-case of (**4**) is used:

$$\begin{gathered} \mathrm{d}s^2 = g_{\mathrm{tt}}(r)c^2\mathrm{d}t^2 - g_{\mathrm{rr}}(r)\mathrm{d}r^2 - r^2(\mathrm{d}\Omega_d)^2, \\ (\mathrm{d}\Omega_d)^2 = \sum_{i=1}^{d}\left[(\mathrm{d}\vartheta^i)^2\prod_{j=1}^{i-1}\sin^2\vartheta^j\right], \end{gathered} \tag{6}$$

here written in higher dimensional spherical coordinates. The particular cases of the last metric are the generalised Schwarzschild solution

$$\mathrm{d}s^2 = \left(1 - \frac{r_g^{d-1}}{r^{d-1}}\right)c^2\mathrm{d}t^2 - \frac{1}{1 - \dfrac{r_g^{d-1}}{r^{d-1}}}\,\mathrm{d}r^2 - r^2\,(\,\mathrm{d}\Omega_d)^2,$$

where

$$r_g^{d-1} = \frac{16\pi}{d}\frac{M}{S_d},$$

with $S_d$ the area of unit $d$-sphere, and for $r \leq r_0$ the perfect fluid solution

$$\mathrm{d}s^2 = \left[\frac{3}{2}\sqrt{1-\frac{r_g}{r_0}} - \frac{1}{2}\sqrt{1-\frac{r_g r^2}{r_0^3}}\right]^2 c^2 \mathrm{d}t^2 - \frac{1}{1-\frac{r_g r^2}{r_0^3}}\,\mathrm{d}r^2 - r^2(\mathrm{d}\Omega_d)^2, \qquad (7)$$

both valid in spacetimes of arbitrary dimension $d+2$. Note that the perfect fluid solution is valid irrespective of dimension.

Within examples, mostly the flat spacetime metrics are used, usually in Cartesian, or spherical coordinates, respectively:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \sum_i (\mathrm{d}x^i)^2 \qquad \mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 - (\mathrm{d}\Omega_d)^2$$

Also, the Reisner-Nordstrom solution

$$\mathrm{d}s^2 = \left(1-\frac{r_g}{r}\right) c^2\mathrm{d}t^2 - \frac{1}{1-\frac{r_g}{r}}\,\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2), \qquad (8)$$

for a charged static black hole has been used. As a general resource for the information on Einstein solutions, [**Ste**] can serve.

A brand new class of physically reasonable solutions to Einstein equations is discovered in chapter ***The Maxwell's fish-eye and gravitational lensing***.

# Part One
# Covariant Optics

The first block of the work deals with establishing the *covariant optics*. We choose this designation for the true optics treatment of *geometrical optics* extension into curved spacetimes. For optics, of highest interest is the focusing and hence, while in general relativity the smallest obstructions are connected with obtaining the optical information finding coordinate-separated solutions to the covariant *eikonal equation* (**3**) (when such solutions exist, of course), this equation will form the basis of our study.

On the other hand, the same optical information is covered by the wavefronts knowledge, we shall present constructions, that utilise them too. The wavefronts can be theoretically obtained from the ray systems from the last paragraph by means of *Legendre transformation*. However, within general relativity, it is even in the simplest cases disabled by mathematical obstructions.

**Summarised, the eikonals, wavefronts and caustics are sought for in this first part of the work, and, particularly, the formulas are developed, which connect the objects mentioned.**

First, eikonal equation is studied from mathematical point of view as a PDE, which gives natural rise to the existence of distinct 'types' of eikonals. Afterwards, the main two complementary types of eikonals are studied with their consequences: the coordinate representation eikonals and the momentum representation eikonals, or shortly the *coordinate eikonals* and *momentum eikonals*, respectively. The point, where mathematical difficulties arise with switching from one to the other representation – by means of Legendre transformation – is pointed out.

Consequently, the *wavefronts family* and *rays family* are explored as basic objects of coordinate and momentum representations, respectively. Continuing to solve the mathematical problems, the *candidates* for these objects are involved and treated.

As a final solution to problems with changing the representation in use, the eikonal Laplaceans are obtained. Namely, the Laplacean of coordinate eikonal is found generally, using momentum representation formulas only.

At last, this Laplacean is shown to be in direct connection with caustic, which in turn shows to be a constant Laplacean of momentum eikonal. Also, these formulas are connected with intensity transfer via optical scalars and the relation of curvature of wavefronts to these matters is discussed.

## Eikonal Equation as a PDE

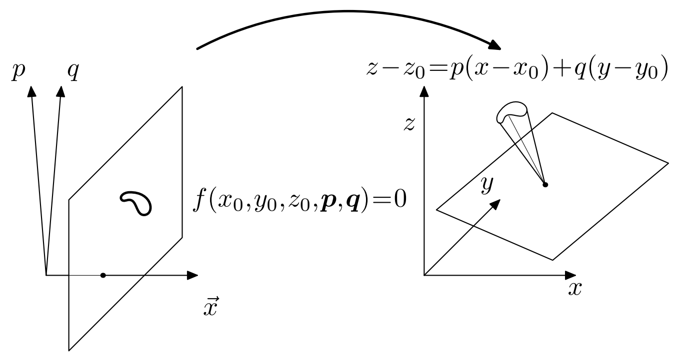
Let us study the (generally nonlinear) partial differential equations of the first order, restricted to two dimensional basis $(x, y)$:

$$f(x, y, z(x, y), p \equiv z_x, q \equiv z_y) = 0 . \qquad (\mathbf{9})$$

First of all, mathematically, the PDE according to (**9**) is not concerned by the physical content of the coordinates used. In a (five-dimensional) space of independent geometrical variables $(x, y, z, p, q)$ the elements

$$\boldsymbol{f(x_0, y_0, z_0, p, q) = 0}$$

constitute an object of codimension one, namely, for $(x_0, y_0, z_0)$ fixed, a curve in the (two dimensional) space of momenta. With the physical meaning of the variables $(p, q)$ it is a set of $(x_0, y_0, z_0)$-located vectors.




The planes to which these vectors are perpendicular are given by $\boldsymbol{z - z_0 = p(x - x_0) + q(y - y_0)}$. As the last equation is one-parametric, these (point-fixed) planes shape a formation; its envelope is a conical surface – the *surface of Monge*.

The equation of one-parameter family $\phi(x, y, \lambda)$ of curves envelope is given by $\phi = \mathbf{0} \wedge \phi_\lambda = 0$. Using parameterisation $(p(\lambda), q(\lambda))$ we obtain

$$\phi_\lambda : p'(x - x_0) + q'(y - y_0) = 0$$

for the planes considered. The solution of this system of (linear) equations in non-degenerate case $p'q - pq' \neq 0$ reads

$$x - x_0 = \frac{q'}{p'q - pq'}(z - z_0) \qquad y - y_0 = \frac{p'}{p'q - pq'}(z - z_0),$$

which is the parametric form of a (part of a) cone, while the cone is defined as a set of lines going through a fixed point and given (closed) curve not containing that fixed point. Such definition is really satisfied by the last expressions: with $z - z_0$ fixed, the right-hand sides form the parametric expression of the cone profile, which is upon freeing $z - z_0$ consequently but vertically scaled. (This cone is of course a double one, concretised by the PDE form.)

In this way, a PDE can be given an interpretation of space field of (Monge) cones. Its solution then, is every surface, that touches the local cones. Such surfaces are called *integral surfaces*.

If re-parametrised to $(p, q(p))$, from implicit derivative formulas we obtain the cone expression as

$$x - x_0 = -\frac{f_p}{q f_q + p f_p}(z - z_0) \qquad y - y_0 = \frac{f_q}{q f_q + p f_p}(z - z_0) \; .$$

These cones degenerate for quasi-linear PDE: the numerators are independent then of momenta, for, in suitable notation, the most general quasi-linear equation gets

$$p f_p(x, y, z) + q f_q(x, y, z) = u(x, y, z) \; .$$

Owing to their form, the denumerators are independent of momenta as well, and, hence, the whole envelope embodies with every plane $z{=}z_0$ the only cross-point – the envelope gets (generally oriented) line. Should the quasi-linear PDE be homogeneous, changing the procedure but slightly, the similar result would be obtained, this time showing the (line) envelope horizontal specially. In both these cases, the solution to PDE chosen is unique.

The eikonal equation in two dimensions is a special case within the family of PDEs followed: when considering a space projection, we obtain purely quasi-quadratic non-homogeneous PDE of the first order, i.e. an equation of a form

$$a(x,y)p^2 + 2b(x,y)pq + c(x,y)q^2 = u(x,y), \qquad (\mathbf{10})$$

with generally non-degenerate coefficients. The cone for this equation, e.g. in parametrisations $(p,\ q(p))$, reads

$$u(x_0,y_0)(x - x_0) = -2(ap + bq)(z - z_0) \qquad u(x_0,y_0)(y - y_0) = 2(cq + bp)(z - z_0) \; .$$

Eikonal equation projection (**10**) – which can be rewrited as $(ap{+}bq)^2 - (b^2 - ac)q^2{=}au$ – degenerates (into two quasi-linear ones) in case $b^2{=}ac$: after such substitution, parameterisation of the profile is independent of momenta.

Note, that the singular integral, given by $f{=}0 \wedge f_p{=}0 \wedge f_q{=}0$ is in non-degenerate case of a form $p{=}q{=}0$ if $u{=}0$ (in full, not time projected case, this corresponds to trivial eikonal $\psi{=}0$), and it exists not, otherwise. In degenerate case, we obtain $ap+bq=0$ if $u=0$, and it exists not, otherwise.

The non-uniqueness of PDR solution can then be interpreted such, that the space field of cones can be connected by integral surfaces touching the cones from 'different sides'. Physically, this is driven by type of constants present in the solution. For eikonal equation, we will in further pick two main cases, namely, when eikonal contains only momentum constants and, when eikonal contains only coordinate constants. Corresponding eikonals we shall label *momentum eikonals* and *coordinate eikonals*, respectively.

A special case occurs in one-dimensional case, when the Monge cones field reduces into field of lines – such can be connected in but a single way. Then, all eikonals (particularly the momentum and coordinate ones) merge, and the eikonal equation has a unique solution (up to signs and additive constant).

*As an illustration, consider the spherical wave in two dimensions. A coordinate eikonal (apart the additive constant) reads*

$$\psi = \omega t \pm \omega\sqrt{r^2 + r_0^2 - 2rr_0\cos(\varphi - \varphi_0)}$$

*and the momentum eikonal under same circumstances reads (as can be in this simple case easily verified using the Legendre transformation)*

$$\psi = \omega t \pm \int \sqrt{\omega^2 - \frac{\Psi^2}{r^2}}\,\mathrm{d}r \pm \Psi\varphi\ .$$

*If coordinate $\varphi$ is suppressed, both these solutions reduce to*

$$\psi = \omega t \pm \omega(r - r_0),$$

*which indeed is the solution to one dimensional eikonal equation. It is worth a note, that to achieve last formulas, formally, both $\Psi=0$ and $\varphi=\varphi_0$ must be set, together with partial usage of integration constant in the second case (which of course does not restrict the additive one).*

Generally, the eikonal equation on a line,

$$g^{\mathrm{tt}}(\psi_{,\mathrm{t}})^2 - 2g^{\mathrm{tx}}\psi_{,\mathrm{t}}\psi_{,\mathrm{x}} - g^{\mathrm{xx}}(\psi_{,\mathrm{x}})^2 = 0,$$

can be in stationary cases separated using $\psi_{,\mathrm{t}}=\omega$, which yields solutions

$$\psi = \omega t - \omega \int \frac{g^{\mathrm{tx}} \pm \sqrt{(g^{\mathrm{tx}})^2 + g^{\mathrm{tt}}g^{\mathrm{xx}}}}{g^{\mathrm{xx}}}\,\mathrm{d}x\ .$$

Note, that the constant $\omega$ becomes automatically a multiplicative one, this behaviour is generic. Also note, that the double signs, occurring in example above, are valid only in static case, i.e. when $g^{\mathrm{tx}}=0$ is added hereby. As the signs before separated parts of momentum eikonal rule the orientation of a ray in that coordinate, it is seen, that the directions of a ray in each coordinate are no longer arbitrary in stationary cases.

**Eikonals As Complete Integrals**

By a *complete integral* of eikonal equation in $(n+1)$-dimensional spacetime we understand a function $\psi(x^k, c_j)$ with $k,\ j=1,\ldots,n$ such, that

$$\psi^{;k}\psi_{;k} = 0 \;\Big|\; \det\left(\frac{\partial^2\psi}{\partial x^\gamma \partial c_\delta}\right) \neq 0 \tag{11}$$

i.e. the rank of its Hessian is maximal. This maximality allows to re-label the constants $c_\delta \to b_\delta(c_\gamma)$, if only the number of degrees of freedom within the constants does not drop. As a consequence, the succession among constant is (mathematically) irrelevant, or – in another of simplest cases – it allows

$$c_1, \ldots, c_\eta \rightarrow b \cdot f(c_2, \ldots c_\eta), c_2, \ldots, c_\eta \ . \tag{12}$$

This particular transformation will be used later. A complete integral, though covering a small class of PDE solutions only, theoretically allows to obtain the *general solution* of a PDE, that covers all the PDE solutions, with generally the only exception of its singular integral, which is not a special case to general solution.

The *singular integral* $\psi_s$ for general eikonal equation (**3**) is yielded from $\partial(\psi_{,j}\psi^{,j})/\partial\psi_{,j}=0$, which gives $\psi^{,j}=0$. Hence, the singular integral reads $\psi_s$=const and we can thus conclude, that from physical point of view, all solutions to eikonal equation are given by any of its complete integrals, while the constant solution is of no relevance.

The (bi)characteristics of eikonal equation

$$\frac{\partial\psi}{\partial c_\gamma} = \text{const}^\gamma$$

constitute the *ray equations* canonically connected with the *wavefronts* $\psi$**=const**. As for the wavefronts, note, that the above eikonal is defined between arbitrary (two) points, however, physically, the phase change is constrained along the rays. For non-singular momentum eikonal $\psi(x^i, c_j)$, the time projection of eikonal equation characteristics

$$\boldsymbol{\pi^j = 0,\ \pi^j \equiv \partial\psi/\partial c_j - \text{const}^j} \tag{13}$$

forms a (hyper)surface of codimension $n-1$ within $(2n-1)$-dimensional space. This hypersurface – thus of dimension $n$ – is designated *enhanced ray (hyper)surface*. The *wavefronts* defined on this surface in addition by $\psi$**=const** are thus of codimension $n$, i.e. are of dimension $n-1$. The rays themselves are always one-dimensional objects.

Apart the formal possibility of Legendre transformation carrying the momentum and coordinate eikonals between each other the last system of equations also formally allows to translate the eikonal into one in *ray coordinates*, where it contains physical information as well. The meaning of ray coordinates is such, that in eikonal, only parameters labelling the rays within their family and one parameter to pick the position along the rays are present. While such eikonals emerge only upon using ray equations, the correct phase change is guaranteed: the number of space coordinates is left so low, that any ray can connect any two points chosen by them. Such an eikonal can also be called an *eikonal along ray*.

*Consider the (momentum) eikonals*

$$\psi - \psi_0 = \omega(t - t_0) - (-1)^m k(x - x_0) - (-1)^l \sqrt{\omega^2 - k^2}(y - y_0) .$$

*The ray equation gets*

$$\frac{\partial}{\partial k} : \qquad -(-1)^m(x - x_0) + \frac{(-1)^l k}{\sqrt{\omega^2 - k^2}}(y - y_0) = 0,$$

*whence – to actually perform the Legendre transformation to coordinate eikonals – we could find the value of parameter $k$,*

$$k = (-1)^u \omega(x - x_0) / \sqrt{(x - x_0)^2 + (y - y_0)^2} .$$

*As non-equivalent modification was performed, the sign (ruled by value of integer constant $u$) has to be precised by check in the ray equation, giving*

$$\mathrm{sgn}(x - x_0) = (-1)^u(-1)^{m+l}\mathrm{sgn}(y - y_0) .$$

*Plugging the expression for $k$ into eikonal, we obtain*

$$\psi = \omega(t - t_0) - \omega\frac{(-1)^{m+u}(x - x_0)|x - x_0| + (-1)^l(y - y_0)|y - y_0|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}},$$

*which, using a small trick, can be written as*

$$\psi = \omega(t - t_0) - \omega\frac{(-1)^{m+u}(x - x_0)^2\frac{|x-x_0|}{x-x_0} + (-1)^l(y - y_0)^2\frac{|y-y_0|}{y-y_0}}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} .$$

*Plugging now calibration of $u$, we finally obtain*

$$\psi = \omega(t - t_0) - \omega(-1)^l \mathrm{sgn}(y - y_0)\sqrt{(x - x_0)^2 + (y - y_0)^2} .$$

*We could however use the ray equation to except the coordinate $y$ from the eikonal:*

$$(y - y_0) = (-1)^{m+l}\frac{\sqrt{\omega^2 - k^2}}{k}(x - x_0) .$$

*Then, we would obtain*

$$\psi = \omega(t - t_0) - (-1)^m\frac{\omega^2}{k}(x - x_0) .$$

*The pair $(k, x)$ now forms the ray coordinates, for $k$ describes which of the rays is considered (looking at ray equation, $\sqrt{\omega^2 - k^2}/k$ is its tangent,* $\tan\phi$*) and $x$ states where on this* *particular ray we are. Note, that the second term in last equation is indeed correct phase difference, while the eikonal can be rewritten as*

$$\psi = \omega(t - t_0) - (-1)^m \omega \frac{\omega}{k}(x - x_0)\ . \ = \omega(t - t_0) - (-1)^m \omega \frac{x - x_0}{\sin\phi}\ .$$

*where the last fraction is actually the distance from the source. Note, that we shall repeat the procedure listed above several times during the work, without necessity thus to go then into this detail.*

A *caustic*, defined as a subset of enhanced ray surface points in which the second is stationary with respect to the subset of parameters, i.e.

$$|\kappa^{jk}| = 0,\ \kappa^{jk} \equiv \partial\pi^j/\partial c_k = \partial^2\psi/(\partial c_j \partial c_k), \qquad (14)$$

is thus of codimension $n$, similarly to wavefronts. However, unlike the rays and wavefronts, that both are $(n-1)$-parametric families, the caustic is $(n-2)$-parametric. This is most plain to see in two dimensional spaces: a (single) ray projections family parameter becomes a parameterisation of caustic, which thus remains parameterless. In this case a single caustic holds all the optical information for the system as will be shown later, hence the benefit if describing the system by means of the caustic.

Let us now have an eikonal $\psi(x^i, c_j)$, its characteristics $\pi^j = 0$, and caustic $|\kappa^{jk}| = 0$. Consider a point (canonical) transformation $\psi(x^i, c_j) \to \tilde{\psi}(x^i, C_j)$. In new variables,

$$\tilde{\pi}^j = \frac{\partial\tilde{\psi}}{\partial C_j} - \mathrm{Const}^j, \qquad \tilde{\kappa}^{jk} = \frac{\partial^2\tilde{\psi}}{\partial C_j \partial C_k}$$

While $\mathrm{Const}^j = \mathrm{const}^i \partial c_i/\partial C_j$, in old variables there is

$$\tilde{\pi}^j = \frac{\partial\psi}{\partial c_l}\frac{\partial c_l}{C_j} - \mathrm{Const}^j$$

and particularly

$$\tilde{\kappa}^{jk} = \frac{\partial^2\psi}{\partial c_i \partial c_l}\frac{\partial c_l}{C_j}\frac{\partial c_i}{C_k} + \frac{\partial\psi}{\partial c_l}\frac{\partial^2 c_l}{\partial C_j \partial C_k} - \mathrm{const}^l \frac{\partial^2 c_l}{\partial C_j \partial C_k} = \frac{\partial^2\psi}{\partial c_i \partial c_l}\frac{\partial c_l}{C_j}\frac{\partial c_i}{C_k}$$

and thus $|\tilde{\kappa}^{jk}| = |\kappa^{jk}||\partial c^i/\partial C^l|^2$. It is seen, that caustics will in new variables emerge there and only there, where in the old ones $\Leftrightarrow$ the transformation considered is diffeomorphism. On the other hand, shall a transformation $P_i(p_j)$ be canonical, its generating function ought to be of a form $F = q^i p_i - Q^i P_i$; then, to hold the canonicity, $q^i = Q^j \partial P_j/\partial p_i$ has to hold. As $\partial P_j/\partial p_i = \partial C_j/\partial c_i$ in our case, it is the property of $C_i(c_j)$ being diffeomorphic that guarantees it is in the same time a canonical transformation, and the canonicity on the other hand implies that $C_i(c_j)$ is diffeomorphism. Summarised, **the caustic is canonical invariant of eikonal**, at least in case of point transformations of eikonal constants.

Note, that apart an additive constant, the solutions to eikonal equation are also left untouched by a multiplicative constant. We can use this fact if we – using (**12**) – succeed to re-label the constants within eikonal in such a way, that one of them becomes purely multiplicative. Then, ray equation for this constant gives no further income, while it reads $\psi$**=const**.

In static spacetimes, on such circumstances, we can equivalently transit to study of the space projection of eikonal only, i.e. to study the wavefronts. Moreover, the space and spacetime Laplaceans of eikonal coincide and also, they coincide with Laplacean of wavefronts up to this multiplicative constant.

*Let us now evaluate the Laplaceans of the well known solutions in the flat spacetime. First of all, in Cartesian coordinates, one momentum solution is*

$$\psi_1 = \omega t - k(x - x_0) - \sqrt{\omega^2 - k^2}(y - y_0),$$

*i.e.* $\Delta\psi_1 = 0$. *The solution is however not unique – a coordinate eikonal*

$$\psi_2 = \omega t - \omega\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

*can be also found, whence* $\Delta\psi_2 = -\omega/\sqrt{(x-x_0)^2+(y-y_0)^2}$. *In spherical coordinates, a coordinate eikonal*

$$\psi_3 = \omega t - \omega\sqrt{r^2 + r_0^2 - 2rr_0\cos(\varphi - \varphi_0)}$$

*exists, whence, by straightforward calculation,* $\Delta\psi_3 = -\omega/\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}$, *which only demonstrates, that the geometrical content of the last two solutions is the same. Note the role of multiplicative constant. The connection between these Laplaceans and wave intensity course (as suggestive from the form of Laplaceans) shall be discussed later on.*

## Eikonal, Rays and Caustic

Let there be an $N+2$ dimensional static $N$-spherically symmetric solution to Einstein equations, valid in spacetime region $\Sigma$. In spherical coordinates $(r,\ \vartheta^i)$, $i=1,2,\ldots,N$, this generally admits metric

$$\begin{aligned}\Sigma\!: \mathrm{d}s^2 &= g_{\mathrm{tt}}(r)\,c^2\mathrm{d}t^2 - g_{\mathrm{rr}}(r)\,\mathrm{d}r^2 - r^2(\mathrm{d}\Omega_N)^2,\\ (\mathrm{d}\Omega_N)^2 &= \sum_{i=1}^{N}\Big[(\mathrm{d}\vartheta^i)^2\prod_{j=1}^{i-1}\sin^2\vartheta^j\Big],\end{aligned} \qquad (\mathbf{15})$$

with $c$ the speed of light. On a non-empty intersection $\sigma$ with equatorial (hyper)surface $\imath<N\!:\ \vartheta^{\imath}={}^{\pi}\!/_{2}$ this brings

$$\Sigma|_{\vartheta^{\imath}={}^{\pi}\!/_{2}} = \sigma:\ \mathrm{d}s^2 = g_{\mathrm{tt}}\,c^2\mathrm{d}t^2 - g_{\mathrm{rr}}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\varphi^2, \qquad \varphi\equiv\vartheta^N\ .$$

In the rest of the work we will restrict ourselves to this cross-section (as will be shown later, this section is actual very general case to needs of optics). There, solutions to eikonal equation $\psi^{,k}\psi_{,k}=0$ of a form

$$\Theta_\sigma:\psi-\psi_0=\frac{\omega}{c}t-(-1)^k\mathrm{p}_\varphi\varphi-(-1)^m\int\sqrt{\frac{\omega^2}{c^2}\frac{g_{\mathrm{rr}}}{g_{\mathrm{tt}}}-\mathrm{p}_\varphi^2\frac{g_{\mathrm{rr}}}{r^2}}\,\mathrm{d}r \qquad (\mathbf{16})$$

are for $k$, $m$ integers its complete integrals (**11**), while their Hessian is non-zero. Consequently, (**16**) may serve as a (momentum) eikonal. Individual terms in previous equation change sign, whenever change in the direction of appropriate coordinate takes place along the path studied. Denote $a(\mathrm{p}_\varphi)$ the value(s) of root under last square root. Due to positive definiteness of metric coefficient(s) – as introduced in (**15**) – this yields

$$a(\mathrm{p}_\varphi):\ \frac{\omega^2}{c^2 g_{\mathrm{tt}}(a)}-\frac{\mathrm{p}_\varphi^2}{a^2}=0\ . \qquad (\mathbf{17})$$

The particular eikonal realising a testing field point source at $[r_s,\ \varphi_s]$ is then

$$\begin{aligned}\theta_\sigma\!:\psi(t,r,\varphi)-\psi_0(t_0,r_s,\varphi_s)&=\frac{\omega}{c}\big(t-t_0\big)-(-1)^k\mathrm{p}_\varphi\big(\varphi-\varphi_s\big)-\\ &-(-1)^m\int_{r_s}^{a}\mp\int_{a}^{r}\sqrt{g_{\mathrm{rr}}(y)\left(\frac{\omega^2}{c^2 g_{\mathrm{tt}}(y)}-\frac{\mathrm{p}_\varphi^2}{y^2}\right)}\,\mathrm{d}y\ ;\end{aligned} \qquad (\mathbf{18})$$

the sign minus or plus between the two integrals depends on whether $r=a$ on the ray lies between the considered end-points of the ray, or not, respectively. The previous expression is valid for regular points $a$ either, so it is valid for all points within considered segment of ray.

Following now the canonical treatment, the particular ray equation is then $\pi_\sigma = \partial\theta_\sigma / \partial p_\varphi$, upon using the Liebnitz rule yielding

$$\pi_\sigma : \varphi - \varphi_s = (-1)^{k+m} \int_{r_s}^{a} \mp \int_{a}^{r} \frac{p_\varphi \sqrt{g_{rr}}}{y^2 \sqrt{\frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{y^2}}} \, dy - (-1)^{k+m} \left( \frac{\partial a}{\partial p_\varphi} \pm \frac{\partial a}{\partial p_\varphi} \right) \left[ \sqrt{g_{rr} \left( \frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{y^2} \right)} \right]_{y=r_0} \tag{19}$$

which – thanks to the defining property (**17**) of rootial points and (parameter) constantness of regular end-points – for all points $a$ turns into

$$\pi_\sigma : \varphi - \varphi_s = (-1)^{k+m} \int_{r_s}^{a} \mp \int_{a}^{r} \frac{p_\varphi \sqrt{g_{rr}}}{y^2 \sqrt{\frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{y^2}}} \, dy \, . \tag{20}$$

The annihilation of last term in (**19**) has the interpretation such, that despite of acquired discontinuity of integrand in rootial points, there is no discontinuity of ray itself in any end-point. Also, it is seen that

$$\left. \frac{dr}{d\varphi} \right|_{r=a} = 0,$$

hence as long as $a$ is a root of odd multiplicity, $r{=}a$ gets clear meaning of turning point on a ray: here, the ray radial coordinate difference must change sign to keep the square-rooted term non-negative for the ray to continue past this point.

The Legendre transformation to carry the transit to coordinate eikonal would mean to except $p_\varphi$ from (**18**) and (**20**). That, unfortunately, is not generally analytically possible. However, it is simple to except $\varphi - \varphi_s$ upon what one obtains an eikonal along ray – the object closest to wavefront(s) description, that is generally available:

$$\lambda_\sigma : \psi_{p_\varphi} - \psi_0 = \frac{\omega}{c}(t - t_0) - (-1)^m \int_{r_s}^{a} \mp \int_{a}^{r} \frac{\omega^2 \sqrt{g_{rr}}}{c^2 g_{tt} \sqrt{\frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{y^2}}} \, dy \, . \tag{21}$$

Obtaining the caustic $\kappa_\sigma = \partial\pi_\sigma / \partial p_\varphi$ is not as straightforward as that of ray equation (**19**), for now there remains a dependence of integral rootial end-points on a derivation parameter, but discontinuity in the integrand in the same end-points is added. This precludes use of Liebnitz rule - the way how to proceed general calculation is to remove the parameter dependence of the end-point(s). This can be done separately in the two integrals of (**20**) by transformations

$$\xi_1 = (y - r_s)/(a - r_s) \qquad \xi_2 = (y - a)/(r - a);$$

differentiating the ray equation after transformations and consequently returning to original variables we finally obtain a caustic

$$\kappa_\sigma:\ 0 = \int_{r_s}^{a} \frac{g_{tt}(\omega^2 y^2 - c^2 g_{tt} p_\varphi{}^2)\left[2g_{rr}^2(\frac{\partial a}{\partial p_\varphi} p_\varphi - a + r_s) + \frac{\partial a}{\partial p_\varphi} p_\varphi (y - r_s)(g'_{rr} - 4g_{rr}^2)\right] + p_\varphi g_{rr}^2\left[2c^2 g_{tt}^2 p_\varphi y(a - r_s) + \frac{\partial a}{\partial p_\varphi}(y - r_s)(\omega^2 g'_{tt} y^3 + 2c^2 g_{tt}^2 p_\varphi{}^2)\right]}{2(a - r_s)\sqrt{g_{rr}}^3 g_{tt}^2 y^5 \sqrt{\frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{y^2}}^3} dy \mp$$
$$\mp \int_{a}^{r} \frac{g_{tt}(\omega^2 y^2 - c^2 g_{tt} p_\varphi{}^2)\left[2g_{rr}^2(\frac{\partial a}{\partial p_\varphi} p_\varphi - a + r) + \frac{\partial a}{\partial p_\varphi} p_\varphi (y - r)(g'_{rr} - 4g_{rr}^2)\right] + p_\varphi g_{rr}^2\left[2c^2 g_{tt}^2 p_\varphi y(a - r) + \frac{\partial a}{\partial p_\varphi}(y - r)(\omega^2 g'_{tt} y^3 + 2c^2 g_{tt}^2 p_\varphi{}^2)\right]}{2(a - r)\sqrt{g_{rr}}^3 g_{tt}^2 y^5 \sqrt{\frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{y^2}}^3} dy\ . \qquad (22)$$

(with prime meaning differentiation according to radial coordinate) if only last integrals converge uniformly. The transformations used were linear; other approaches are possible, e.g. transformations of type $y = a \pm \xi^2$ would remove the singularity of integrands in turning end-points. Of course, when there are no turning points present within the ray segment under consideration, the caustic from (**16**) is simply

$$0 = \int \frac{\omega^2 \sqrt{g_{rr}}}{c^2 g_{tt}} \frac{dr}{\sqrt{\frac{\omega^2}{c^2 g_{tt}} - \frac{p_\varphi^2}{r^2}}^3}\ . \qquad (23)$$

*As an example consider Minkowski spacetime, which can be covered by a single metric*

$$\mathbf{R}^2 \times \mathbf{S}^2 = \Sigma :\ ds^2 = c^2 dt^2 - dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2\vartheta\, d\varphi^2\ .$$

*According to (**20**), the ray equation on equatorial section is in such a case*

$$\pi_\sigma : \varphi - \varphi_s = (-1)^{k+m} \int_{r_s}^{a} \mp \int_{a}^{r} \frac{p_\varphi\, dy}{y^2} \Big/ \sqrt{\frac{\omega^2}{c^2} - \frac{p_\varphi^2}{y^2}}\ .$$

*To be able to profit from the general construction developed, let us prevent ourselves from evaluating the last integral. In that way, at last, the caustic as (**22**) gets*

$$\kappa_\sigma : (-1)^{k+m}\left(\frac{1}{\sqrt{r^2-\varrho^2}} \pm \frac{1}{\sqrt{{r_s}^2-\varrho^2}}\right) = 0,$$

*with* $\varrho=(p_\varphi c)/\omega$ *non-negative without loss of generality. It is now clearly seen that before the ray turning point of* $r=\varrho$ *as from (**17**), there lies the only caustic point – the source itself at* $r=r_s$ *(the second coordinate* $\varphi=\varphi_s$ *is obtained from the ray equation stated above in this example). After turning point, there are no caustic points at all (in correspondence with the beams constant divergence). Also, for the calculation presented,* $r\geq\varrho$ *has to hold. In this way, in the flat case,* $\varrho$ *has directly the meaning of ray closest advance point towards origin (turning point) radial coordinate. In further, we stick to this notation and will label the rays by* $\varrho$*. Also note, that by introduction of* $\varrho$*, the constant* $\omega$ *turned multiplicative.*

### Parametric derivatives of singularity containing integrals

To generalise the method of obtaining the caustic (**22**), let us now study the calculation of parametric (Riemann) integrals

$$I \equiv \int_{a(\varrho)}^{b(\varrho)} f(y,\varrho)\,\mathrm{d}y \qquad (24)$$

containing singularity(ies) of integrand within integration path as well as the calculation of their derivative with respect to parameter – such integrals often occur within ray equations, when turning points are present, and their derivative is of interest when looking for caustics. Of main interest to us is the derivative $\partial I/\partial\varrho$ general construction without necessity to evaluate the integral itself at the beginning of calculation, for in cases of interest, such evaluation is seldom available. The task to do of course is to transform the integral so, that general derivative lemmas may be applied.

*Lemma.* Let the function $f(y,\varrho)$ together with its partial derivative $\partial f/\partial\varrho$ be continuous within $a\leq y\leq b$, $\varrho_0\leq\varrho\leq\varrho_1$. Then, for $\varrho_0\leq\varrho\leq\varrho_1$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\int_a^b f(y,\varrho)\,\mathrm{d}y = \int_a^b \frac{\partial}{\partial\varrho} f(y,\varrho)\,\mathrm{d}y\ .$$

*Lemma. (Liebnitz rule)* In addition to presumptions of previous lemma, let the functions $u(\varrho)$, $b(\varrho)$ be differentiable within $\varrho_0\leq\varrho\leq\varrho_1$ and $a\leq u(\varrho)\leq b$, $a\leq v(\varrho)\leq b$ holds within that interval. Then, for $\varrho_0\leq\varrho\leq\varrho_1$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\int_{u(\varrho)}^{v(\varrho)} f(y,\varrho)\,\mathrm{d}y = \int_{u(\varrho)}^{v(\varrho)} \frac{\partial}{\partial\varrho} f(y,\varrho)\,\mathrm{d}y + f(v(\varrho),\varrho)\frac{\mathrm{d}v(\varrho)}{\mathrm{d}\varrho} - f(u(\varrho),\varrho)\frac{\mathrm{d}u(\varrho)}{\mathrm{d}\varrho}$$

The first Lemma holds for improper integrals either, if it is assumed that

$$\int\limits_a^b f(y,\varrho)\,\mathrm{d}y \quad \text{converges, and that} \quad \int\limits_a^b \frac{\partial}{\partial\varrho} f(y,\varrho)\,\mathrm{d}y$$

converges uniformly within $\varrho_0 \le \varrho \le \varrho_1$ (in such a case, the function $f(y,\varrho)$ and its derivative $\partial f/\partial\varrho$ are considered continuous only within $a < y \le b$, $\varrho_0 \le \varrho \le \varrho_1$ or within $a \le y < b$, $\varrho_0 \le \varrho \le \varrho_1$). The integrand end-point possible discontinuity is the crucial difference of the two Lemmas.

One of our possibilities is thus to remove the end-point (general) parameter dependence using parameter-dependent transformation

$$h : y = h(\xi) \,\Big|\, \frac{\partial h}{\partial\varrho} \neq 0,\ \frac{\mathrm{d}}{\mathrm{d}\varrho} h^{-1}(b) = \frac{\mathrm{d}}{\mathrm{d}\varrho} h^{-1}(a) = 0\ .$$

In this way, we arrive to the diagram

$$\begin{array}{ccc} I = \int_{a(\varrho)}^{b(\varrho)} f(\varrho,y)\mathrm{d}y & \xrightarrow{\quad \boldsymbol{h^*} \quad} & \int_{h^{-1}(a)}^{h^{-1}(b)} (h\circ f)h'\mathrm{d}\xi \\ \Big\downarrow \boldsymbol{\frac{\partial}{\partial\varrho}} & & \Big\downarrow \boldsymbol{(h^{-1})^*\frac{\partial}{\partial\varrho}} \\ \frac{\partial I}{\partial\varrho} & \xrightarrow{\quad \mathbf{id} \quad} & \int\limits_{a(\varrho)}^{b(\varrho)} \left(h^{-1}\circ\left[h'\frac{\partial}{\partial\varrho}(h\circ f) + (h\circ f)\frac{\partial h'}{\partial\varrho}\right]\right)(h^{-1})'\mathrm{d}y \end{array} \qquad \textbf{(25)}$$

in which the left column has the meaning of calculating the derivative on evaluated integral.

Let then now the integral (**24**) exist. As $I$ is Riemann integral, the set of singular points of its integrand is of measure zero, hence the integral can be split into sum of integrals with singularities of integrands in end-point(s) $s$ only. If the individual integrals in the sum exist (which is guaranteed in the case that the only singularity is lying in the end-point of $I$ by the existence of full integral (as is the common case in this work)), we can due to their common behaviour devote our attention without loss of generality to a single integral

$$J = \lim_{\lambda\to s(\varrho)^\pm} \int\limits_{a(\varrho)}^{\lambda} f(y,\varrho)\,\mathrm{d}y,$$

where the approach direction in the limit is chosen to be from inside of the integration path. As a particular choice,

$$h : y = (s-a)\xi + a\ .$$

if sufficient. While it is independent of integrand, it may serve as general tool for studying the diagram (**25**). Carrying the transformation, we have

$$J = \lim_{\lambda\to 1^-}\int_0^\lambda f(\varrho, (s-a)\xi + a)(s-a)\,\mathrm{d}\xi\ .$$

and after derivating and returning to original variable we, using the diagram (**25**), have

$$\frac{\partial J}{\partial\varrho} = \lim_{\lambda\to s}\int_a^\lambda \left\{\frac{\partial f}{\partial\varrho} + \frac{\partial a}{\partial\varrho} f' + \left[(y-a)f\right]'\frac{\partial}{\partial\varrho}\ln(s-a)\right\}\mathrm{d}y, \tag{26}$$

where prime stands for partial derivative with respect to $y$. Here the general procedure must be stopped, for we generally cannot presume the existence of separate limits if split last integral. It is however clearly seen, that for integrals containing not singularities, the last two terms of integrand could be integrated out and the Liebnitz rule be obtained.

*Let us demonstrate the acquired results on a single example: let*

$$J = \int_{r_0}^{\varrho}\frac{\varrho}{y\sqrt{y^2-\varrho^2}}\,\mathrm{d}y,$$

*then* $\varphi - \varphi_s = J$ *is the ray equation in the flat case. Evaluation of* $J$ *is simple, and using the direct way, one would obtain*

$$\frac{\partial J}{\partial\varrho} = 1\Big/\sqrt{r_0^2-\varrho^2}, \tag{a}$$

*let us however use the scheme derived. The general formula (**26**) gives, term by term*

$$\frac{\partial J}{\partial\varrho} = \lim_{\lambda\to\varrho}\int_{r_0}^{\lambda}\left\{\frac{y}{\sqrt{y^2-\varrho^2}^{\,3}} + 0 + \frac{\varrho(-2y^2r_0 + r_0\varrho^2 + y^3)}{y^2(y^2-\varrho^2)(r_0-\varrho)\sqrt{y^2-\varrho^2}}\right\}\mathrm{d}y, \tag{b}$$

*and indeed, the calculation of*

$$\lim_{\lambda\to\varrho}\int_{r_0}^{\lambda}\frac{r_0(\varrho^2+\varrho y - y^2)}{(\varrho-r_0)(y+\varrho)y^2\sqrt{y^2-\varrho^2}}\,\mathrm{d}y$$

*e.g. using partial fractions, yields the correct result (**a**). It is worth a note, that the limits of individual integrals in (**b**) do indeed not exist, as anticipated during general derivation.*

Let us finally applicate the results acquired to the case of eikonals, i.e. the situation with double signs:

$$\tilde{I} = \int_{r_0(\varrho)}^{a(\varrho)} \mp \int_{a(\varrho)}^{r_1(\varrho)} f(\varrho, y)\,\mathrm{d}y\ .$$

Let there the nature of the problem studied imply the impossibility to reach the region $r < a(\varrho)$, i.e. let $r_0,\, r_1 \geq a(\varrho)$. In other words, $a(\varrho)$ shall be the single turning point, an infimum of radial coordinate rays can reach. Recall, that in case of eikonals, the upper sign corresponds to the case of end-points separated on the ray by a turning point, and the lower one, when not. In the latter case, however, insertion of turning point is artificial and we are interested, whether the formulas found would then reduce to usual derivative of integral with respect to its parameter, while there are no singularities. (In case $r_0{=}r_1$ this reduction is, of course, trivial.)

If the integral is to exist, then there has to exist a limit in the last expression. Then, also a Cauchy principal value for the integral exists, moreover, the equality of all these values is guaranteed. For $a, b > \varrho$ we can thus generally write

$$\tilde{I} = \lim_{\varepsilon\to 0^+} \left\{ \int_{r_0(\varrho)}^{a(\varrho)+\varepsilon} \pm \int_{a(\varrho)+\varepsilon}^{r_1(\varrho)} f(\varrho, y)\,\mathrm{d}y \right\} .$$

If we are now to transform $\tilde{I}$, we have to use

$$h_0 : y = (s + \varepsilon - r_0)\xi_1 + r_0 \quad \text{a} \quad h_1 : y = (s + \varepsilon - r_1)\xi_1 + r_1 .$$

With these transformations we have

$$\lim_{\varepsilon\to 0^+} \left( \int_0^1 f(\varrho, (s+\varepsilon-r_0)\xi_0+r_0)(s + \varepsilon - r_0)\,\mathrm{d}\xi_0 \pm \int_1^0 f(\varrho, (s+\varepsilon-r_1)\xi_1+r_1)(s + \varepsilon - r_1)\,\mathrm{d}\xi_1 \right)$$

and, after carrying out,

$$\frac{\partial \tilde{I}}{\partial \varrho} = \lim_{\varepsilon\to 0^+} \left( \int_{r_0}^{s+\varepsilon} \left[ \frac{\partial f}{\partial \varrho} + \frac{s' - r_0'}{s+\varepsilon-r_0} [(y-r_0)f]' + \frac{\partial f}{\partial y} r_0' \right] \mathrm{d}y \pm \right.$$
$$\left. \pm \int_{s+\varepsilon}^{r_1} \left[ \frac{\partial f}{\partial \varrho} + \frac{s' - r_+'}{s+\varepsilon-r_1} [(y-r_1)f]' + \frac{\partial f}{\partial y} r_1' \right] \mathrm{d}y \right) .$$

While with these end-points the integrands are not singular, we can owing to order of limit and integration finally write

$$\lim_{\varepsilon\to 0^+} \left( \int_{r_0}^{s+\varepsilon} \pm \int_{s+\varepsilon}^{r_1} \frac{\partial f}{\partial \varrho}\,\mathrm{d}y + f(\varrho, s+\varepsilon)[s' \mp s'] - r_0' f(\varrho, r_0) \pm r_1' f(\varrho, r_1) \right) .$$

This is already the sought for result, while in the case of end-points not separated by turning point, the upper sign is valid, which cancels the singular terms in front of the square bracket before the application of limit and we would arrive to Liebnitz rule.

## Wavefronts and Rays Candidates

In general relativity, there is a strict distinction between rays and wavefronts in the sense that, due to mathematical limitations, it is usually not analytically possible to switch from one description to the other, even though we know, that these two one-parametric families are locally perpendicular to each other.

Hence, to avoid the Legendre transformation, it is of great value to be able to find connective formulas, that enable us to use single representation quantities during whole computation. Namely to this task the following chapters are devoted.

A problem similar to previous one, is that in general relativity we usually don't have the 'degree of freedom' to manipulate the forms of the rays when acquired at all. This sort of problems lies basically in the fact, that many transformations of the family equations are describing the same geometrical objects, but do not have the same physical qualities (e.g. do not fulfil eikonal equation). We thus have to generally distinguish very sharply the objects (rays, wavefronts) from just *candidates* for these objects within general relativity. We shall adopt this distinction right from this very beginning to avoid later confusion.

A good place to start with is the equation, stating that wavefronts are locally transversal to rays. Actually – to catch a glimpse of forthcoming computations – if we for a moment restrict ourselves to the flat case of two dimensions, we can say, that wavefronts and rays families are an example of *isogonal lines*, i.e. the families of plane curves, whose members are everywhere (locally) incident with constant angle:

> *Let $\phi(x,y,\lambda)$ be a one parameter family of curves. Then, the family of curves which are always locally inclined by angle $\gamma$ to curves of $\phi$ is given by*
>
> $$\left(\frac{\partial\phi}{\partial x}\cos\gamma - \frac{\partial\phi}{\partial y}\sin\gamma\right)\mathrm{d}x + \left(\frac{\partial\phi}{\partial x}\sin\gamma + \frac{\partial\phi}{\partial y}\cos\gamma\right)\mathrm{d}y = 0\,.$$

Particularly, for transversal families $\gamma = {}^{\pi}/_{2}$ and we obtain

$$-\frac{\partial\phi}{\partial y}\,\mathrm{d}x + \frac{\partial\phi}{\partial x}\,\mathrm{d}y = 0\,.$$

Having the structure of last formula in mind, one can easily follow the idea of general construction presented further.

**Wavefronts**

Let $\boldsymbol{h}$**=const** be the projection of wavefronts in stationary spacetime $(M,\mathrm{g})$, i.e. let a coordinate eikonal $\psi=\omega t-\omega h$ fulfil the eikonal equation $\mathrm{g}(\mathrm{d}\psi,\ \mathrm{d}\psi)=0$. Note, that eikonal equation is insensitive to diffeomorphisms of eikonals. Indeed, when $m{:}\mathbf{R}^1\to\mathbf{R}^1$, for $\tilde\psi=m\circ\psi$ one obtains $\mathrm{g}(\mathrm{d}\tilde\psi,\ \mathrm{d}\tilde\psi)=m'^2\mathrm{g}(\mathrm{d}\psi,\ \mathrm{d}\psi)$.

Let then a codimension one one-parameter family $\bar f$ of (hyper)curves be a candidate for a wavefront:

$$\bar{\boldsymbol{f}}(\boldsymbol{x^a},\bar{\mathbf{c}}) = \mathbf{0} \Rightarrow \bar{\boldsymbol{h}}(\boldsymbol{x^a}) = \bar{\mathbf{c}}, \tag{27}$$

with $\bar h$ not necessarily existing explicitly. Though $\mathrm{g}(\mathrm{d}\bar\psi,\ \mathrm{d}\bar\psi)=0$ is for $\bar\psi=\omega t-\omega\bar h$ sufficient condition for $\bar h$ to be a wavefront, it is not the condition necessary: making use of the fact that diffeomorphisms of eikonals do not affect the eikonal equation we have to also admit a diffeomorphism $c\circ\bar h$ of the wavefronts candidates, yielding for the true coordinate eikonals $\psi=\omega t-\omega(\mathrm{c}\circ\bar h)$ an eikonal equation

$$\frac{1}{\omega^2}\mathrm{g}(\mathrm{d}\psi,\ \mathrm{d}\psi) = g^{tt} - 2c' g^{\mathrm{t}a}\bar h_{,a} - (c')^2 g^{ab}\bar h_{,a}\bar h_{,b} = 0, \tag{28}$$

where $a,\ b$ are space indices. The last equation will be fulfilled identically, if

$$\mathrm{c} = \int \frac{\partial\bar f}{\partial\bar{\mathrm{c}}}\,\frac{g^{\mathrm{t}a}\bar f_{,a} \pm \sqrt{\left(g^{\mathrm{t}a}\bar f_{,a}\right)^2 + g^{\mathrm{tt}}g^{ab}\bar f_{,a}\bar f_{,b}}}{g^{ab}\bar f_{,a}\bar f_{,b}}\,\mathrm{d}\bar{\mathrm{c}} \tag{29}$$

while $\bar h_{,a}=-\bar f_{,a}/(\partial\bar f/\partial\bar{\mathrm{c}})$, using implicit derivatives formulas. A candidate (**27**) is thus a wavefront if we happen to find last integral, namely if the integrand can be made rid of the coordinates (using (**27**) again), while as long as $c(\bar c)$ is function of a single variable, it holds no place for coordinates. In this way, using the freedom of eikonal transformation, we can precise the diffeomorphism for all candidates, that are indeed wavefronts.

*Let us investigate two wavefronts candidates* $\bar f^{\pm}$: $\mathbf{sin(3}\boldsymbol{x}\mathbf{+5}\boldsymbol{y}\mathbf{)+\bar c cos(3}\boldsymbol{x}\mathbf{\pm 5}\boldsymbol{y}\mathbf{)=0}$ *in two dimensions: plugging into (***29***) we get*

$$|\mathrm{c}| = \int \frac{\partial\bar f^{\pm}/\partial\bar{\mathrm{c}}}{\sqrt{(\bar f^{\pm}_x)^2 + (\bar f^{\pm}_y)^2}}\,\mathrm{d}\bar{\mathrm{c}} = \int \frac{\cos(3x \pm 5y)}{\sqrt{3^2+5^2}[\cos(3x+5y) - \bar{\mathrm{c}}\sin(3x \pm 5y)]}\,\mathrm{d}\bar{\mathrm{c}}$$

*and using the definition of* $\bar f^{\pm}$ *we finally get*

$$|c| = \frac{1}{\sqrt{24}}\int \frac{\mathrm{d}\bar{\mathrm{c}}}{\dfrac{\cos(3x+5y)}{\cos(3x\pm 5y)} + \bar{\mathrm{c}}^2} \quad \begin{cases} \mathrm{c}\nexists\ \textit{for } f^- \\ \mathrm{c}=\pm\dfrac{1}{\sqrt{24}}\arctan\bar{\mathrm{c}}+\mathrm{const}\ \textit{for } f^+ \end{cases}.$$

*Indeed $h=(3x+5y)/\sqrt{24}$ is a wavefront for $f^+$, i.e. it satisfies eikonal equation.*

Let us now devote our attention to the case of $Y$-parametrised form of wavefront candidate

$$\bar{f}^a(x^a, Y, \bar{C}) = 0 \Rightarrow Y^{(a)} = \bar{g}^a(x^a, \bar{c})$$

with again, $\bar{g}$ not necessarily existing explicitly. In last formula, $Y^{(a)}$ means (possibly hypothetic only) expression of $Y$ from $a$-th equation, i.e. also $Y^{(a)} \equiv Y^{(b)}$. Then an implicit (non-parametric) expression for a wavefront candidate $\bar{f}(x^i, \bar{c})=0$ is any of $\bar{f}(Y^{(a)})=\bar{f}(\bar{g}^a(x^a,\bar{c}))$ where $\bar{f}$ means just balancing the $Y^{(a)}$s to vanish. For such a candidate expression to be directly plugged into (**29**) we only need to state the partial derivatives:

$$\bar{f}_{,a}=\frac{\partial \bar{f}}{\partial \bar{g}^b}\bar{g}^b_{,a}=\frac{\partial \bar{f}}{\partial \bar{g}^b}Y^{(b)}_{,a}=-\sum_b \bar{f}^b_{,a}\frac{\partial \bar{f}}{\partial \bar{g}^b}\Big/\frac{\partial \bar{f}^b}{\partial Y}$$

$$\frac{\partial \bar{f}}{\partial \bar{c}}=\frac{\partial \bar{f}}{\partial \bar{g}^b}\frac{\partial \bar{g}^b}{\partial \bar{c}}=\frac{\partial \bar{f}}{\partial \bar{g}^b}\frac{\partial Y^{(b)}}{\partial \bar{c}}=-\sum_b \frac{\partial \bar{f}^b}{\partial \bar{c}}\frac{\partial \bar{f}}{\partial \bar{g}^b}\Big/\frac{\partial \bar{f}^b}{\partial Y}$$

The most simple case is for two dimensions: $\bar{f}^a(x^a, Y, \bar{C})=0$, $a=1,2$. Choosing obvious $\bar{f}=Y^{(1)}-Y^{(2)}$ we have

$$\bar{f}_{,1} = -\bar{f}^1_{,1}\Big/\frac{\partial \bar{f}^1}{\partial Y}, \qquad \bar{f}_{,2} = \bar{f}^2_{,2}\Big/\frac{\partial \bar{f}^2}{\partial Y}, \qquad \frac{\partial \bar{f}}{\partial \bar{c}} = -\frac{\partial \bar{f}^1}{\partial \bar{c}}\Big/\frac{\partial \bar{f}^1}{\partial Y}+\frac{\partial \bar{f}^2}{\partial \bar{c}}\Big/\frac{\partial \bar{f}^2}{\partial Y}$$

These expressions can now be directly plugged into explicit case formulas.

**Rays**

Let $f$ be a function over Riemannean manifold $(M,\mathrm{g})$ of dimension $m$. Then $\boldsymbol{f(x^i)=c}$ is a solution to equation $\mathbf{d}\boldsymbol{f}\mathbf{=0}$ and represents itself a codimension one hypersurface. For $g\colon R\to R$, $\boldsymbol{g\circ f=\tilde{c}}$ is a solution to the same equation $\Leftrightarrow$ $\boldsymbol{f=c}$ is a solution, if only $g$ is a diffeomorphism.

Indeed, in coordinates one has $\mathrm{d}f=f_{,i}\mathrm{d}x^i$, $\mathrm{d}c=0$. Furthermore, $\mathrm{d}(g\circ f)=g' f_{,i}\mathrm{d}x^i=g'\mathrm{d}f$. Note also, that $\tilde{c}=g(c)$ for both solutions to be identical.

Let now $m=2$; then over M there exist functions α, $h$ such, that $\alpha{*}\,\mathrm{d}f=\mathrm{d}h$. As a consequence, the gradients of $f$ and $h$ are perpendicular.

Actually, $*\mathrm{d}f$ is an $(m-1)$-form, with $\alpha\not\equiv 0$ being its integration factor (if such exists). For $m=2$ both $\mathrm{d}f$ and $*\mathrm{d}f$ are one forms. If we pick coordinates $x$, $y$ on $M$, then

$*\mathrm{d}f=(1/\sqrt{g})f_{,\mathrm{y}}\mathrm{d}x-(1/\sqrt{g})f_{,\mathrm{x}}\mathrm{d}y$ and, comparing the components, $h_{,\mathrm{x}}=(\alpha/\sqrt{g})f_{,\mathrm{y}}$, $h_{,\mathrm{y}}=-(\alpha/\sqrt{g})f_{,\mathrm{x}}$. Excepting $h$ from the last pair of equations by taking the cross derivatives one obtains

$$\frac{1}{\sqrt{g}}\left(\alpha_{,\mathrm{x}}f_{,\mathrm{x}}+\alpha_{,\mathrm{y}}f_{,\mathrm{y}}\right)+\alpha\left[\left(\frac{f_{,\mathrm{x}}}{\sqrt{g}}\right)_{,\mathrm{x}}+\left(\frac{f_{,\mathrm{y}}}{\sqrt{g}}\right)_{,\mathrm{y}}\right]=0 \tag{30}$$

by whose solvability, existence of $\alpha$ in two dimensions is guaranteed.
Also, $\mathrm{g}(\mathrm{d}f,\mathrm{d}h)=\alpha\mathrm{g}(\mathrm{d}f,*\mathrm{d}f)=\alpha\varepsilon^{ij}f_{,i}f_{,j}=0$.

The construction of $f$ and $h$ is chosen such, that they might serve as ray and wavefront family candidates, respectively. For this reason, let $h$ be further a solution to space projection $\omega^2\mathrm{g}(\mathrm{d}h,\mathrm{d}h)=\omega^2 g^{\mathrm{tt}}$ of eikonal equation:

$$g^{ij}h_{,i}h_{,j}=g^{xx}\alpha^2 f_{,\mathrm{x}}^2+2\alpha^2 g^{xy}f_{,\mathrm{x}}f_{,\mathrm{y}}+g^{yy}\alpha^2 f_{,\mathrm{y}}^2=g^{\mathrm{tt}}$$

which yields $\alpha^2=g^{\mathrm{tt}}/g^{ij}f_{,i}f_{,j}$. While the integration factor is a multiplicative quantity, we will without loss of generality use the positive root of previous equation,

$$\alpha=\sqrt{g^{\mathrm{tt}}/g^{ij}f_{,i}f_{,j}}, \tag{31}$$

moreover for – as will be shown – we are mainly interested in cases when $\alpha\to 0$.

One would find (**30**) not identically fulfilled upon introduction of last expression. Instead, upon substituting therein for $\alpha$ from (**31**) a requirement for $f$ to be an momentum eikonal candidate is acquired:

$$\begin{aligned}2g^{ij}f_{,i}f_{,j}\left[(\ln\sqrt{g^{\mathrm{tt}}})_{,x}f_{,\mathrm{x}}+(\ln\sqrt{g^{\mathrm{tt}}})_{,y}f_{,\mathrm{y}}+\sqrt{g}\left(\frac{f_{,\mathrm{x}}}{\sqrt{g}}\right)_{,x}+\sqrt{g}\left(\frac{f_{,\mathrm{y}}}{\sqrt{g}}\right)_{,y}\right]-\\-\left[(g^{ij}f_{,i}f_{,j})_{,\mathrm{x}}f_{,\mathrm{x}}+(g^{ij}f_{,i}f_{,j})_{,\mathrm{y}}f_{,\mathrm{y}}\right]=0\end{aligned} \tag{32}$$

However, $\alpha$ is multiplicative quantity, which can be used to gain a significant mathematical simplification of the procedure that must be taken: in practise, it is better to find $\alpha$ from (**31**) and subsequently checked there is to be (**32**); if (**32**) gets broken, the input function $f$ may be a system of trajectories, but never admissible by eikonal equation, i.e. not rays in physical meaning as objects in geometrical optics. In this way, **the checking of rays candidate has been reduced to derivating only**.

Clearly, if $\alpha\equiv 1$, $f$ forms itself a coordinate eikonal.

*If chosen $M\equiv\mathbf{R}^2$ together with Cartesian coordinates, the equation (**32**) reduces to $f_{,\mathrm{x}}^2 f_{,\mathrm{yy}}+f_{,\mathrm{y}}^2 f_{,\mathrm{xx}}-2f_{,\mathrm{x}}f_{,\mathrm{y}}f_{,\mathrm{xy}}=0$ and $\alpha^2=1/(f_{,\mathrm{x}}^2+f_{,\mathrm{y}}^2)$.*

*For a one parameter family of candidates* $f = x^2 + y^2 + 2axy$ *the calibration (**31**) is*

$$\alpha = 1 \Big/ \left( 2\sqrt{(x+ay)^2 + (y+ax)^2} \right)$$

*and the condition (**32**) distinguishing rays from common curves gets*

$$-8(1-a^2)(x^2+y^2-2axy) = 0 \, .$$

*Clearly, only systems with* $a = \pm 1$ *fulfil last equation identically irrespective of coordinates, ie. are rays together with all diffeomorphisms of themselves.*

As has been shown in the first paragraph of this chapter, even if $f$ is a diffeomorphic mapping $f = g \circ \phi(x^i)$ of ray system $\phi$, it is still a ray system. As a consequence of (**32**), any among such transformed ray systems fulfils its condition, so **there is no need to further distinguish between rays candidate and physical rays** .

Plugging $f = g \circ \phi$ into (**32**) we indeed obtain

$$g'^3 \left( \begin{array}{r} 2g^{ij}\phi_{,i}\phi_{,j} \left[ (\ln\sqrt{g^{tt}})_{,x}\phi_{,x} + (\ln\sqrt{g^{tt}})_{,y}\phi_{,y} + \sqrt{g}\left(\frac{\phi_{,x}}{\sqrt{g}}\right)_{,x} + \sqrt{g}\left(\frac{\phi_{,y}}{\sqrt{g}}\right)_{,y} \right] - \\ - \left[ (g^{ij}\phi_{,i}\phi_{,j})_{,x}\phi_{,x} + (g^{ij}\phi_{,i}\phi_{,j})_{,y}\phi_{,y} \right] \end{array} \right) = 0$$

which is formally identical with (**32**). Also,

$$\alpha = \sqrt{g^{tt}} \Big/ \left( g' \sqrt{g^{ij}\phi_{,i}\phi_{,j}} \right) \, .$$

Note, that while the integration factor is multiplicative, the mapping-dependence is of no particular importance (until we want to know the wavefronts, of course).

Let us now study the case of implicitly given ray equation, i.e. $F(x^i, c) = 0 \Rightarrow f(x^i) = c$. Of course, $f$ is now expected to not be available for some particular reason.

Using now the implicit derivatives calculus, we can write

$$f_{,i} = -\frac{F_{,i}}{F_c},$$

if considered $F = 0 \Rightarrow f = c$. When obtaining $f_{,ij}$ and higher derivatives, care must be taken, because in last equation, the parameter $c$ is generally present, which should be (prior to further derivating) excluded therefrom using ray equation. We however cannot expect that we can perform the transit from $F$ to $f$, so last derivative must be understood as $f_{,i}(x^i, c(x^i))$. With this in mind, we can compute the second derivative as

$$f_{,ij} = -\frac{\partial}{\partial x^i}\frac{F_{,i}}{F_c} - \frac{\partial}{\partial c}\frac{F_{,i}}{F_c}\frac{\partial c}{\partial x^i}$$

Using the fact that $\boldsymbol{f=c}$ holds we have $c_{,i}=f_{,i}$ in rightmost term of last equation, hence finally

$$f_{,ij} = -\frac{F_{,ij}}{F_c} + \frac{F_{,i}\dfrac{\partial F_{,j}}{\partial c} + F_{,j}\dfrac{\partial F_{,i}}{\partial c}}{F_c^2} - \frac{F_{,i}F_{,j}F_{cc}}{F_c^3}\ .$$

Note that last expression is symmetric in indices $i,\ j$ as partial derivative must be. We have thus obtained the second derivative without need to now $c$ explicitly. Again $c$ is present in last expression, so to find the third derivative, we would have to take $f_{ij}(x^i, c(x^i))$ which yields

$$f_{,ijk}=-\frac{F_{,ijk}}{F_c}+\frac{F_{,(ij}\dfrac{\partial}{\partial c}F_{,k)}+F_{,(i}\dfrac{\partial}{\partial c}F_{,jk)}}{F_c^2}-$$
$$-\frac{F_{cc}F_{,(i}F_{,jk)}+2F_{,(i}\dfrac{\partial}{\partial c}F_{,j}\dfrac{\partial}{\partial c}F_{,k)}+F_{,(i}F_{,j}\dfrac{\partial^2}{\partial c^2}F_{,k)}}{F_c^3}+$$
$$+\frac{3F_{cc}F_{,(i}F_{,j}\dfrac{\partial}{\partial c}F_{,k)}+F_{,i}F_{,j}F_{,k}F_{ccc}}{F_c^4}-3\frac{F_{,i}F_{,j}F_{,k}(F_{cc})^2}{F_c^5},$$

where parentheses in indices mean symmetrization over cyclic permutation of included ones. It is important to point out, that all the derivatives in last expressions are partial, i.e. all the implicit information has been smeared out. The higher order derivatives bring no new ideas.

The expressions from the implicit case can be just plugged into formulas of the implicit one, to obtain the desired result.

As for the parametrically given rays candidates, the treatment is formally the same as has been shown for the parametric case for ***Wavefronts***.

## The Laplaceans of eikonals

Finally, using all the information gathered, we can find the value of (coordinate) Laplacean $\Delta h \equiv \delta \mathrm{d}h$ of the wavefront in $m$-dimensional space:

$$\delta \mathrm{d}h = *\mathrm{d}*\mathrm{d}h = *\mathrm{d}*(\alpha * \mathrm{d}f) = *\mathrm{d}(\alpha ** \mathrm{d}f) = (-1)^{m-1} * \mathrm{d}(\alpha \mathrm{d}f) = \\ = (-1)^{m-1} * (\mathrm{d}\alpha \wedge \mathrm{d}f)$$

while rays are always codimension one. In components,

$$(\Delta h)^{a_1 \dots a_{m-2}} = (-1)^{p(m-p)} \varepsilon^{a_1 \dots a_{m-2} ul} \alpha_u f_{,l} = \\ = (-1)^{(m-1)} \varepsilon^{a_1 \dots a_{m-2} ul} \frac{\alpha^3}{2 g_{\mathrm{tt}}} (g^{kv} f_{,k} f_{,v})_{,u} f_{,l} \ .$$

Expanding the bracket, we can write

$$(\Delta h)^{a_1 \dots a_{m-2}} = \frac{(-1)^{m-1}}{2 g_{\mathrm{tt}} \sqrt{g^{mn} f_{,m} f_{,n}}^3} \varepsilon^{a_1 \dots a_{m-2} ul} f_{,l} (g^{kv}_{,u} f_{,k} f_{,v} + 2 g^{kv} f_{,k} f_{,vu}) \tag{33}$$

The importance of formula (**33**) is that a quantity of a coordinate representation eikonal was generally obtained using the momentum eikonal only. In other words, **we can state the Laplacean of coordinate eikonal generally, if only we know the rays pertaining this eikonal**. In this way, the need of Legendre transformation has been avoided. Note however, that the Laplacean contains now the momentum parameter(s), which would have to be eliminated using the ray equation $\boldsymbol{f(x^i)=c}$, for Laplacean to become a stand alone quantity. This is the price for avoiding the Legendre transformation. As will be seen later this price is not too high.

Consider now the case of implicit ray equation $\boldsymbol{F(x^i, c)=0}$. The Laplacean $\Delta h$ in this case reads

$$(\Delta h)^{a_1 \dots a_{m-2}} = \frac{(-1)^{m-1} \varepsilon^{a_1 \dots a_{m-2} ul}}{2 g_{\mathrm{tt}} \sqrt{g^{mn} F_{,m} F_{,n}}^3 |F_c|} F_{,l} \Bigg[ g^{kv}_{,u} F_{,k} F_{,v} F_c^2 - \\ - 2 g^{kv} F_{,k} \left( -F_{,vu} F_c^2 + \left( F_{,v} \frac{\partial F_u}{\partial c} + F_{,u} \frac{\partial F_v}{\partial c} \right) F_c - F_{,u} F_{,v} F_{cc} \right) \Bigg] \tag{34}$$

Note that formally the explicit ray equation case is restored, when all parameter derivatives of $F_{,i}$ and $F_{cc}$ (and thus all higher terms in both cases) are set zero. This rule is valid generally, not only within Laplacean. In the flat case, the implicit case Laplacean (**33**) reduces to

$$* \mathrm{d} * \mathrm{d}h = \frac{f_{xy}(f_x^2 - f_y^2) - f_x f_y (f_{xx} - f_{yy})}{\sqrt{f_x^2 + f_y^2}^3} \tag{35}$$

and the explicit case Laplacean (**34**) to

$$\Delta h = \Big[F_x F_y \Big(F_c^2(F_{yy}-F_{xx})+2F_c(F_x F_{xc}+F_y F_{yc})-F_{cc}(F_y^2-F_x^2)\Big)+ +(F_y^2-F_x^2)\Big(-F_c^2 F_{xy}+(F_x F_{yc}+F_y F_{xc})F_c-F_x F_y F_{cc}\Big)\Big]\frac{1}{2\sqrt{F_x^2+F_y^2}^3\,|F_c|}$$

*Let us choose the family* $\boldsymbol{F\!:y-y_0=c(x-x_0)}$*of curves, which upon* $[x_0, y_0]$ *fixed represents a bunch of lines through that point. Such a choice represents a ray equation, while the implicit form of (***32***) is fulfilled identically. Also,*

$$\alpha = (x-x_0)\Big/\sqrt{(x-x_0)^2+(y-y_0)^2}$$

*The Laplacean now gets* $\Delta h = -1\Big/\sqrt{(x-x_0)^2+(y-y_0)^2}$*.*

*For the rays of momentum eikonal (apart turning points) we know to have*

$$\boldsymbol{f:\varphi-\varphi_0-\int_{r_0}^{r}\frac{\Psi}{y\sqrt{\omega^2 y^2-\Psi^2}}\,dy=0}$$

*for which the (***32***) is of course fulfilled. Then* $\alpha = \sqrt{\omega^2 r^2-\Psi^2}/\omega$ *and finally*

$$\Delta h = \frac{1}{\sqrt{\omega^2 r_0^2-\Psi^2}-\sqrt{\omega^2 r^2-\Psi^2}} \wedge \Psi = \frac{\omega r r_0\sin(\varphi-\varphi_0)}{\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}}$$

*while the ray equation is invertible in* $\Psi$*. Together,*

$$\Delta h = \frac{\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}}{r_0|r_0-r\cos(\varphi-\varphi_0)|-r|r-r_0\cos(\varphi-\varphi_0)|}$$

*The last expression deserves a piece of commentary: the combination of expressions within absolute brackets is such, that their signs vary from point to point, but always remain opposite from one bracket to the other. This allows us to finally write*

$$\Delta h = -1\Big/\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}\,.$$

*It is another clue, that the two hereby examined examples are geometrically identical. Both the Laplaceans stated are in correspondence with the direct computation in example at the end of* ***Eikonals as complete integrals****.*

Even more important information carries the Laplacean (**34**) in the implicit case. It states, that (under certain regularity conditions) **the implicit case Laplacean of coordinate eikonal diverges at and only at the caustic itself**. In this way, from focusing point of view, the momentum (caustic) and coordinate (Laplacean) treatments are equivalent and any available of them can be in particular application used.

Indeed, recall the equation (**14**) of caustic $|\kappa^{jk}| \equiv |\partial\pi^j/\partial c_k| = 0$. In two dimensional case, the ray equations $\pi^j = 0$ reduce to single equation $F = 0$, and hence the caustic gets $\kappa\!: F_c = 0$.

Looking at equation (**34**), the presence of term $|F_c|$ in the denominator assures the statement of previous paragraph.

Note, that actually, the caustic itself is a parameter-Laplacean of momentum eikonal.

## The Optical Scalars

The actual role of optical scalars is two-fold: first, if the vector field $\xi$ defining a (geodesic) congruence is known in some region of spacetime, the optical scalars are known there as well (by direct computation), which gives us the possibility to qualify the vector field behaviour (focusing etc. ). Second, if the vector field $\xi$ is specified but in a form of boundary values, we can from the affine properties of the background spacetime using optical scalars re-construct the global properties of all admissible vector fields, or sometimes, the fields themselves.

The usual point to start with optical scalars is the acceleration vector, giving at last the optical scalars as quantities within its *kinematical decomposition*. Namely the (effectively three-dimensional) *expansion tensor* $\theta_{ab}$ and *vorticity tensor* $\omega_{ab}$ are introduced. Taking the decomposition to traceless part and the trace, the usual *Sachs scalars* may be introduced:

$$\theta = \frac{1}{2} l^k_{;k} \qquad \omega^2 = \frac{1}{2} l_{[k;m]} l^{k;m} \qquad \theta^2 + |\sigma|^2 = \frac{1}{2} l_{(k;m)} l^{k;m} \,. \tag{36}$$

Technically, these four scalars can be combined into two complex valued ones with nice geometrical interpretation. Using Ricci identity, the relation of these quantities to Riemann tensor is established. Hence these quantities lie in roots of the spacetime structure. Also the general evolution equations for these scalars may be generally obtained – the *Raychaudhuri equation* with its consequences.

Let us now adopt quite different approach, based on purely mathematical ideas. We have the task to describe the behaviour of geodesic congruences tangent vector fields $l^k$ in terms of scalars. To cover the information on local behaviour of such fields, we need to admit the first derivatives $l^{k;j}$ in these scalars. Let us now build a scheme of such scalars, sequently adding the number of vector field components involved within groups with particular number of first derivatives present:

The lowest number of derivatives is zero and there are only scalars of type

$$l^k l_k \,,\ (l^k l_k)^2 \,,\ \ldots$$

where dots mean higher powers of $l^k l_k$. As can be seen, all the information within these zero order is contained in single term $l^k l_k$.

There are two types of scalars containing a single first derivative:

$$l^k_{;k} \,,\ \ldots \qquad l^{k;j} l_k l_j \,,\ \ldots$$

As can be seen, any information other then already known is in a term $l^k_{;k}$, while the second one is identically zero for geodesics.

The situation slightly complicates for two first derivatives present, we now have

$$\begin{aligned}
&l^k_{;k} l^j_{;j} \ldots \\
&l^{k;m} l_{k;m},\ l^{k;m} l_{m;k} \cdots \\
&l^{k;m} l_{j;m} l^j l_k,\ l^{k;m} l_{j;k} l^j l_m,\ l^{m;k} l_{m;j} l^j l_k,\ \ldots \\
&l_{k;m} l_{u;v} l^k l^m l^u l^v,\ \ldots
\end{aligned}$$

(with the summary power of $l^k$ rising with each line). There is no new information on the first line, and the last one trivialises for geodesics. The second line brings both two terms new. The third one is trivial in the last two terms due to geodesicity again, however the first term here trivialises itself for scalar congruences $l^k{=}\psi^{,k}$ too, as then $l^{k;j}{=}l^{j;k}$. However, when scalar congruences are treated, from the same reason only one term of the second line is independent.

As for three first derivatives, we can write

$$\begin{aligned}
&l^k_{;k} l^j_{;j} l^m_{;m},\ l^k_{;j} l^j_{;m} l^m_{;k},\ \ldots \\
&l^k_{;k} l^{m;j} l_{m;j},\ l^k_{;k} l^{j;m} l_{m;j},\ \ldots \\
&l^k_{;k} l^j_{;j} l^{j;m} l_{j;u} l_m l^u,\ \text{many permutations} \ldots \\
&\ \vdots
\end{aligned}$$

The number of terms is now high, however, whole blocks are dependent only: e.g. starting on line three there is a part that could be written as $l^k_{;k}$·*(terms with two first derivatives)* etc. Also, note that from this order on, the Ricci identity comes into play. Even more interestingly, when the scalar congruences are considered, there are no new terms at all from here on.

Summarising, if all the scalar terms that carry independent non-trivial information are typed bold, then all of the bold terms for scalar isotropic geodesic congruences are listed in the scheme

$$\begin{aligned}
\text{none:}\quad & l^k l_k,\ \ldots \\
\text{one:}\quad & \boldsymbol{l^k_{;k}},\ l^{k;j} l_k l_j,\ \ldots \\
\text{two:}\quad & l^k_{;k} l^j_{;j},\ \boldsymbol{l^{k;m} l_{k;m}},\ l^{k;m} l_{m;k},\ \ldots \\
& l^{k;m} l_{j;m} l^j l_k,\ l^{k;m} l_{j;k} l^j l_m,\ \ldots \\
& l_{k;m} l_{u;v} l^k l^m l^u l^v,\ \ldots \\
\vdots\quad &
\end{aligned}$$

It is seen, that the usual Sachs scalars (**36**) for our case are constructed namely from the above basis in the way, they had the well-known physical meaning of expansion, rotation (which vanishes for light), twist and shear, that rule the infinitesimal beam cross-section (and its evolution). Let us study now the optical scalars for light congruences. From caustical viewpoint, the distortion makes no contribution, the rotation is identically zero for gradient cases $l^k = \psi^{,k}$, as is our case. Thus, only expansion remains, given as

$$\theta = \frac{1}{2} l^{\alpha}_{;\alpha} \, .$$

Its double has the physical meaning of relative change to cross-section $A$ of an infinitesimal beam with respect to affine parameter $\sigma$ change, i.e.

$$l^{\alpha}_{;\alpha} = \frac{\mathrm{d}A}{A\,\mathrm{d}\sigma},$$

or, solving last equation

$$\int l^{\alpha}_{;\alpha} \,\mathrm{d}\sigma = C \ln A \, .$$

which gives an answer to where the caustic is to appear: it is everywhere, where the beam cross-section vanishes, i.e. everywhere, where the integral with respect to affine parameter of Laplacean shows negative divergence.

Let us now consider the solution $\psi$ of eikonal equation. As was shown in (**3**), thanks to the scalarity the rays are geodesics. In static cases, the projection of eikonal equation reads $g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} = g^{\mathrm{tt}}\psi_{,\mathrm{t}}^2$. The left-hand side of the equation is then an eikonal equation in space, for in this case the space metric is just the minor of spacetime metric. The right-hand side has the meaning of a wave vector norm (with time independence assured). It is thus seen, that the space projections of rays will be themselves the space (non-isotropic) geodesics $\Leftrightarrow g^{\mathrm{tt}}$=const, for conservation of norm is a property of parallel transport (and $\psi_{,\mathrm{t}}$=const).

When searching for the affine parameter of the rays we thus have to stick to spacetime eikonal equation. Then, however, while the rays are isotropic geodesics, the arc will not be an affine parameter. Yet, while we know, the rays are geodesics, we expect that from the geodesic expression in general parameterisation $v$

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}v^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}v} \frac{\mathrm{d}x^k}{\mathrm{d}v} + \frac{\mathrm{d}x^i}{\mathrm{d}v} \frac{\mathrm{d}^2 v}{\mathrm{d}\sigma^2} \Big/ \left(\frac{\mathrm{d}v}{\mathrm{d}\sigma}\right)^2 = 0 \qquad (\mathbf{37})$$

we will succeed to find the affine parameter $\sigma$. The equations above indeed reduce into one, if a condition

$$\frac{\mathrm{d}^2 x^{[i}}{\mathrm{d}v^2} \frac{\mathrm{d}x^{l]}}{\mathrm{d}v} + \frac{\mathrm{d}x^j}{\mathrm{d}v} \frac{\mathrm{d}x^k}{\mathrm{d}v} \Gamma^{[i}_{jk} (\mathrm{d}x^{l]} / \mathrm{d}v) = 0$$

(which removes from the geodesic equation (**37**) the parameterisation dependence) holds [**Kuch**]. If it does, we can pick for finding the affine parameter any of geodesic equation (**37**) components.

To utilise these formulas, we will in further expect the knowledge of eikonal with multiplicative constant in two dimensions, $\psi = \omega t - \omega\chi(x^i, \rho)$. Its ray equations are given by

$$\partial/\partial\omega : t - t_0 = \chi \qquad \partial/\partial\rho : \chi_\rho = \mathrm{const}$$

As an trial parameter we always pick one of coordinates. For the general treatment, of course, we then choose the component of (**37**) that is the parameter coordinate one, for its simplicity (the second derivatives vanish). Then,

$$\Gamma^{v}_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}v} \frac{\mathrm{d}x^k}{\mathrm{d}v} v_\sigma^2 + v_{\sigma\sigma} = 0 \; .$$

Taking now the implicit derivatives formulas, we can write $v_\sigma = 1/\sigma_v$, $v_{\sigma\sigma} = -\sigma_{vv}/\sigma_v^3$, whence

$$\Gamma^{v}_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}v} \frac{\mathrm{d}x^k}{\mathrm{d}v} \sigma_v - \sigma_{vv} = 0,$$

which finally gives

$$\sigma_v = \exp \int \left( \Gamma^{v}_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}v} \frac{\mathrm{d}x^k}{\mathrm{d}v} \right) \mathrm{d}v \; .$$

There is no need to proceed further this general calculation: upon integration of Laplacean with respect to affine parameter we need not know the parameter itself, for $\mathrm{d}\sigma = \sigma_v \, \mathrm{d}v$ holds. The result stated above is the only thing needed.

It is however interesting, what is to be done with the ray parameter $\varrho$ during the last calculation. As affine parameter is to be a function of trial parameterisation only, and we picked one of coordinates for it, the other one has to be extincted using ray equation, and the ray parameter is to be integrated as a constant this time.

*When starting from ray equations, in Cartesian coordinates of the flat case the equations of geodesics both for plane and spherical waves reduce thanks to vanishing of Christoffel symbols to the fact, that the coordinates themselves are the affine parameters. Things change, when polar coordinates are adopted: let us start from momentum eikonal*

$$\psi = \omega t - \Psi\varphi - \int \sqrt{\omega^2 - \frac{\Psi^2}{r^2}} \, \mathrm{d}r$$

*which brings ray equations*

$$t - t_0 = \int \frac{r \, \mathrm{d}r}{\sqrt{r^2 - \rho^2}} \qquad \varphi - \varphi_0 = \int \frac{\rho \, \mathrm{d}r}{r\sqrt{r^2 - \rho^2}}$$

*after transit to multiplicative constant. As a natural parameterisation of the geodesic we choose the radial coordinate. As expected, carried out, all geodesic equations reduce into single equation*

$$r(r^2 - \rho^2)r_{\sigma\sigma} - \rho^2 r_\sigma^2 = 0\ .$$

*Its solution is simple:*

$$\sigma_r = \exp\left(-\int \frac{\rho^2\,\mathrm{d}r}{r(r^2-\rho^2)}\right) = \frac{r}{\sqrt{r^2-\rho^2}}$$

*whence finally*

$$\sigma = \sqrt{r^2-\rho^2} - \sqrt{r_0^2-\rho^2} = \frac{r|r - r_0\cos(\varphi-\varphi_0)| - r_0|r_0 - r\cos(\varphi-\varphi_0)|}{\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}}$$

*after substituting for* $\varrho$ *from ray equation. Note that actually the affine parameter is the distance from the source of radiation. This is not a surprise, while in the flat case, the projections of geodesics are geodesics, whose affine parameter is of course the arc-length.*

*In connection with this, when in Cartesian coordinates turned the coordinates themselves the affine parameters, then, when choosing a point source, i.e. the rays* $\boldsymbol{y-y_0=k(x-x_0)}$*, let us choose as the parameter e.g.* $(x-x_0)$*. Then, however, a distance from source is also an affine parameter, while* $\sqrt[n]{(x-x_0)^n+(y-y_0)^n}=(x-x_0)\sqrt[n]{1+k^n}$*, after substitution from ray equation.*

## The Curvature Landscape

It is a thoroughly accepted fact, that in the flat (three-dimensional) space the general wavefront exhibits two caustic surfaces: one for each principal curvature of points of itself [**Ber**]. Much more interesting is that the family wavefronts ruled by eikonal equation (**3**) have the property, that all of its members exhibit the identical curvature surfaces, i.e. the curvature surfaces are invariants of wavefront evolution. We show in this chapter, how such behaviour comes into being.

In statements of previous paragraph, there is a connection between curvature and intensity divergence (in rank of geometric optics) contained. In the forthcoming chapter we show how this comes into being and moreover, find the exact connection.

Let us then deal with case $M \equiv R^2$ for a while and state therein the expression for a (first, geodesic) curvature

$$\mathrm{K} = (\dot{x}\ddot{y} - \dot{y}\ddot{x}) / \sqrt{\dot{x}^2 + \dot{y}^2}^{\,3} \tag{38}$$

for a curve $(x(\tau), y(\tau))$. Such parameterisation is not suitable for our purposes, having in mind the wavefronts equation $\boldsymbol{h(x,y)=c}$, it is more convenient to parameterise as $(x, y(x))$. Then,

$$\mathrm{K} = y'' / \sqrt{1 + y'^2}^{\,3} \; .$$

Having the implicit wavefront, we can state by direct computation, that

$$\mathrm{K} = -\frac{(h_y^2 h_{xx} - 2h_x h_y h_{xy} + h_x^2 h_{yy})}{\sqrt{h_x^2 + h_y^2}^{\,3}} \; . \tag{39}$$

The last expression, however, is the Laplacean (**35**). To see it plain, let us plug in the appropriate ray equation $f(x,y)=c$ (taking the explicit one makes no loss of generality), i.e. the connection $(h_x, h_y) = (\alpha f_y, -\alpha f_x)$ with $\alpha = 1/\sqrt{f_x^2 + f_y^2}$. This yields

$$\mathrm{K} = \frac{f_{xy}(f_x^2 - f_y^2) - f_x f_y (f_{xx} - f_{yy})}{\sqrt{f_x^2 + f_y^2}^{\,3}}$$

which coincides with (**35**). We can thus conclude, that (in two flat dimensions) **the Laplacean of coordinate eikonal identifies itself with the wavefronts curvature**.

Now we have in hand all the material to study the relationship of the wavefronts and caustic (at least in the flat case). First of all, we know the caustic to appear in the places of coordinate eikonal Laplacean divergence. So we can move along a ray and watch the value of Laplacean: as we have seen recently, it in the same time acquires the reciprocal value of wavefront radius of curvature at intersection with ray followed. By these two points the change of Laplacean value is tightened such, that we know in every point, that the caustic will appear on the ray followed just the curvature radius farther along this ray (in appropriate direction along the ray, of course). These two points also guarantee, that all wavefronts have the common caustic. The word 'farther' in one but last sentence is used very justly, because, as we have seen, that the affine parameter is just the arc length.

Summing up, while the rays are in the flat case straight lines in our case, to whom the caustic points are always from the local wavefront farthered by the curvature radius along these rays, we can state that enhanced to whole wavefront, **the caustic appears as a geometrical location of wavefront curvature centres, which is the definition of (wavefront) evolute**.

The expression for evolute $e{=}(e_x(\tau),\ e_y(\tau))$ of a curve $(X(\tau), Y(\tau))$ we however do know:

$$e = \left( X - \frac{(\dot{X}^2 + \dot{Y}^2)\dot{Y}}{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}\ ,\ Y + \frac{(\dot{X}^2 + \dot{Y}^2)\dot{X}}{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}} \right),$$

which reparameterised to $y(x)$ gives

$$e = \left( X - y'\frac{1 + y'^2}{y''},\ Y + \frac{1 + y'^2}{y''} \right)\ .$$

Taking again the wavefronts $\boldsymbol{h(x,y)=c}$ we obtain

$$e = \left( X - \frac{h_x^2 + h_y^2}{h_y^2 h_{xx} - 2h_x h_y h_{xy} + h_x^2 h_{yy}} h_x,\ Y - \frac{h_x^2 + h_y^2}{h_y^2 h_{xx} - 2h_x h_y h_{xy} + h_x^2 h_{yy}} h_y \right),$$

which can be re-written using (**39**) as

$$e = \left( X + \frac{1}{\mathrm{K}} \frac{h_x}{\sqrt{h_x^2 + h_y^2}},\ Y + \frac{1}{\mathrm{K}} \frac{h_y}{\sqrt{h_x^2 + h_y^2}} \right)\ .$$

Plugging again $(h_x, h_y){=}(\alpha f_y, -\alpha f_x)$ we finally obtain

$$e = \left( X + \alpha f_y/\mathrm{K},\ Y - \alpha f_x/\mathrm{K} \right)\ .$$

Here the computations are exhausted, because the final relationship (between wavefronts and caustic) was found. However, we insist to not know the components $X, Y$ defining the wavefront, when the ray equation is known for our calculation to be of a value; the only thing we know, is that $X,\ Y$ obey the equation of wavefront, i.e. $\boldsymbol{h(X,Y)=c}$, whence, for the relationship discussed, finally

$$\boldsymbol{h(x - \alpha f_y/\kappa, y + \alpha f_x/\kappa, c) = 0 \ \wedge\ f = 0}\ .$$

Note that in the last expression for evolute (caustic) the second equation serves to extinct the ray parameter. In this way, caustic (unlike one-parameter families of rays and wavefronts) indeed turns non-parametric in the two-dimensional case, i.e. there is a single curve to describe the optical situation all properties.

While we insist to not know the explicit form of wavefronts, we will not even tray to extend this last calculation in the curved case, instead, we adopt in ***Part Two*** of this work another approach, that will provide us with further results.

For the purpose of completeness, let us at this end state the calculation of the curve involute – we would make us of it, whenever we know explicitly the caustic and will seek for the wavefronts:

$$\nu : \left( X - s\dot{X} \big/ \sqrt{\dot{X}^2 + \dot{Y}^2}\,,\ Y - s\dot{Y} \big/ \sqrt{\dot{X}^2 + \dot{Y}^2} \right), \qquad (\mathbf{40})$$

where $s = \int \sqrt{\dot{X}^2 + \dot{Y}^2}\,\mathrm{d}t$. Namely from this (indefinite) integral emerges the constant, that is to be a wavefronts family parameter.

# Exercise & Applications

This part of the work contains a simple exercise and several applications of the treatment from the main two parts of this work.

**Axial beam reflection on a spherical mirror**. This exercise is mostly a demonstration of majority of derivations performed within this work. Aimed was an application on some flat configuration, that is however a bit more sophisticated then the plane and spherical waves, that are used for their immediate illustrativity of the problematic during the calculations. Thus, here, the case of beam whose rays come (without loss of generality) parallel with symmetry axis is shown collected on a single place. For this reason, the maximum of references to points, where each appropriate problem is discussed, is given.

**The higher-dimensional optics**. Though the calculations of the first part of this work were done mainly in two dimensions, it is shown in this short application, that their validity is not restricted by this number of dimensions. Note, that an even more generalising insight of two dimensional optics region of validity is discussed in the last part of the work.

**The Maxwell's fish-eye and gravitational lensing**. There are only few physical realisations of an ideal optical instrument – apart from the flat mirror, the Maxwell's fish-eye is one. In this application a gravitational lensing configuration is found, which in particular limit reproduces this ideal optical instrument.

**The focus of a cluster and its aberrations**. Here the optical constructions built are fully applied to gravitational lensing problem: a bending of point source light by intervening matter of foreground cluster of galaxies possible realisation. The value of cluster focal length is given with its aberrational structure. Physical discussion is present.

**The light within FLRW models**. In this last application, the formulas are tested on a metric, that is very different from ones, used during the main calculations to prove the versatility of the methods suggested.

## Axial Beam Reflection on a Spherical Mirror

Let us (without loss of generality) consider a section of unit mirror $X^2+Y^2=1$, that is located at origin of Cartesian coordinates within the flat spacetime, as can be seen in Fig. 1. We choose the rays from distant source and, allowed by the symmetry of the problem, we let the beams impact along the optical axis (chosen as horizontal).

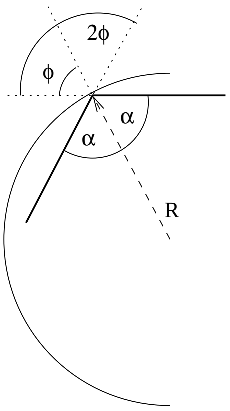



Fig. 1. *The situation at the symmetry section of spherical mirror. One ray from the incident beam (approaching from right) is shown together with this ray reflection. The point of reflection is* $[-\sqrt{1-Y^2},\, Y]$.

As rays in Minkowski spacetime are formed by lines, they can be generally written as $y=p_Y x+q_Y$, with a ray point on the mirror coordinate $Y$ chosen as a (unique) ray parameter, for the family of y-axis parallel rays. For the single-time reflected rays this (using the mirror equation) brings

$$\boldsymbol{y - Y = k(x + \sqrt{1 - Y^2})}\ .$$

It only remains to find the directive of the reflected ray. Noticing that the reflection normal line is in any point of the mirror a radial one, we can from the Snell's law write

$$k = \tan -2\phi = -\tan 2 \arctan \frac{Y}{\sqrt{1-Y^2}} = -2Y \frac{\sqrt{1-Y^2}}{1-2Y^2} \tag{41}$$

and the sought for *ray equation* is finally

$$\boldsymbol{\pi : (1 - 2Y^2)(y - Y) = -2Y\sqrt{1-Y^2}(x + \sqrt{1-Y^2})}, \tag{42}$$

here adjusted to allow also for the reflected rays without directive, profiting now the fact, that we can work with implicit systems, as developed in chapter ***Rays***. From all possible non-singular (linear) parameterisations we choose

$$\begin{aligned} X_\pi &= (1-2Y^2)\tau - \sqrt{1-Y^2} \\ Y_\pi &= -2Y\sqrt{1-Y^2}\,\tau + Y \end{aligned} \tag{43}$$

which is most quickly seen from the ray derivative (**41**) and a (natural) choice, that the parameter-origin was a mirror reflection point on a ray.

The *caustic*, as a singularity of *enhanced ray surface* appropriate to (**42**) projection onto the configuration space $\partial\pi/\partial Y=0$ emerges as

$$\kappa : (1-2Y^2)^2 = 4Y^4 - 2Y^2 + \frac{2x}{\sqrt{1-Y^2}} + 2 \tag{44}$$

The equations (**42**) and (**44**) settle the parametric form of caustic

$$\begin{aligned} X_\kappa &= -(1/2+Y^2)\sqrt{1-Y^2} \\ Y_\kappa &= Y^3 \ . \end{aligned} \tag{45}$$

Note how the role of ray family parameter $Y$ changed to parameterisation of caustic, see (**14**) and further. The caustic itself is from (**45**) seen to be of the semi-cubic (Neil) parabola type, i.e. a symmetric curve possessing a cusp singularity, see Fig. 2. The caustic cusp, as point lying on the optical axis is thus defined as $Y=0$ within (**45**), which gives $x|_{Y=0}=-{}^1\!/_2$, which is in agreement with the well-known relation $f=R/2$ for the focal length of a spherical mirror with radius $R$.

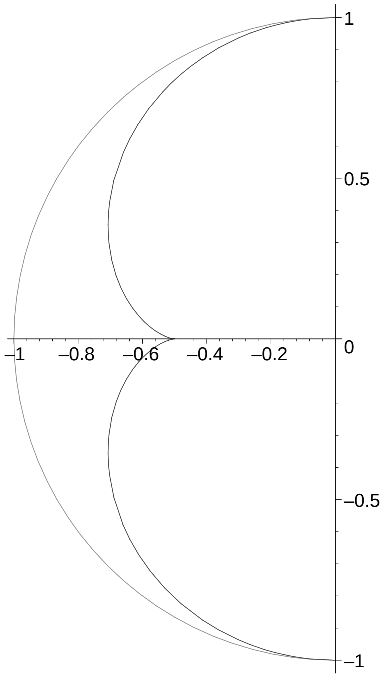



Fig. 2. *The caustic (***45***) of rays single-time reflected on a spherical mirror symmetry section. Note the position of the cusp.*

The *wavefronts* cannot be found from the ray equation (**42**), unless we stated the appropriate *momentum eikonal*, moreover, due to algebraic problems, analytically at best with problems.

On the other hand, we have shown for the flat case, that the wavefronts are formed by the *involute* (**40**) of caustic; in our case

$$X_w = -2(1-Y^2)\sqrt{1-Y^2} + C(1-2Y^2)$$
$$Y_w = -2Y^3 + 3Y - 2YC\sqrt{1-Y^2} \qquad (46)$$

with the constant $C$ – coming from the indefinite integral within involute expression – numerating the wavefronts; e.g. $C=1$ is the one, that touches the mirror horizontal pole.Note, that it is however not obvious, that $C$ was the optical length (i.e. the affine parameter) itself.

Having a wavefronts parameterisation (**46**), we can easily find their curvature (**38**) as

$$\kappa = (\dot{X}_w \ddot{Y}_w - \dot{Y}_w \ddot{X}_w) \Big/ \sqrt{\dot{X}_w^2 + \dot{Y}_w^2}^{\,3} \; .$$

The actual result for our exercise is quite complicated, however, stating the values of curvature along the axial ray is illustrative:

$$\kappa|_{Y=0} = -2/|2C - 3| \; . \qquad (47)$$

According to last formula, at the axis, the wavefronts curvature diverges at $C={}^3\!/_2$, which occurs (from the wavefront equation (**46**)) at $x=-{}^1\!/_2$. Owing to symmetry, this point is the cusp of the caustic present and also, it is the value of position of focus of the mirror used (see Fig. 2.). Such result is in correspondence with (**45**).

Let us draw our attention back to wavefronts. Having now their parametric form (**46**), the previous suspicion that obtaining their explicit form $h(x,y)=c$ would be difficult is confirmed, for although eliminating the parameter $Y$ is a matter of solving algebraic equations of third and second order (most easily for $\sqrt{1-Y^2}$ from the first equation, using a trick $1-2Y^2=2(1-Y^2)-1$), the trial for wavefront explicit formulas by eliminating the parameter $C$ must be considered problematic at best right from the beginning.

In strict contrast to complications of the last paragraph, there stands the easiness of obtaining the depiction of wavefronts by aberration expansion, using the general wavefront expansion (**81**) in the flat spacetime with Cartesian coordinates

$$h = (x + c_0) + \frac{1}{2x + c_2} y^2 + \frac{-2x + c_4}{(2x + c_2)^4} y^4 \; . \; . \; . \; . \qquad (48)$$

To comply with the last expansion, we must first of all check, whether the formulas in use are just candidates, or more, while they came ad hoc in the beginning. Within the $\mathbf{R}^2$ considered we according to (**28**) obtain

$$0 = \bar{c}'^2 \left( \frac{\partial x}{\partial \bar{c}} \Big/ \frac{\partial x}{\partial Y} - \frac{\partial y}{\partial \bar{c}} \Big/ \frac{\partial y}{\partial Y} \right)^2 - \frac{1}{(\partial x/\partial Y)^2} - \frac{1}{(\partial y/\partial Y)^2} \; .$$

After substituting, we get $c'=1$, which means, that the candidate constitutes directly the wavefronts. Hence, their parameter $C$ is affine, i.e. the arc in the flat case, and as late as now the statement avoided above, that $C$ measures the optical length becomes just. We can thus directly substitute the wavefront parameterisation (**46**) into general expansion (**48**), obtaining

$$\left(-2(1-Y^2)^{3/2}+C(1-2Y^2)+c_0\right)+\left(\frac{(-2Y^3+3Y-2YC\sqrt{1-Y^2})^2}{-4(1-Y^2)^{3/2}+2C(1-2Y^2)+c_2}\right)+$$
$$+\left(\frac{4(1-Y^2)^{3/2}-2C(1-2Y^2)+c_4}{(-4(1-Y^2)^{3/2}+2C(1-2Y^2)+c_2)^4}(-2Y^3+3Y-2YC\sqrt{1-Y^2})^4\right)+\ldots=C\ .$$

Re-expanding in powers of $Y$, we get

$$(-2+c_0)+\left(3-2C+\frac{(3-2C)^2}{-4+c_2+2C}\right)Y^2+$$
$$+\left(-3/4+\frac{2(3-2C)(C-2)}{-4+c_2+2C}-\frac{2(3-2C)^3}{(-4+c_2+2C)^2}+\frac{(4+c_4-2C)(3-2C)^4}{(-4+c_2+2C)^4}\right)Y^4+\ldots=0\ .$$

Note, in what manner this calculation is meaningful: every next term contributes to one order higher term than the one before, and we can thus recursively get

$$c_0=2,\qquad c_2=1,\qquad c_4=-\frac{5}{4},\ \ldots$$

Also note, that the constant $C$ numerating the wavefronts, does not appear in the expansion coefficients, it only appears in the wave progress difference (**86**).

To conclude these calculations, we shall once more show the location of the Gaussian focus, using now the aberrations treatment. For two general axial wavefronts, we can (in a fixed point) always match their progress by a suitable choice of phase differences (equalising the lowest order constants in their expansions), see (**87**). The aberration coefficients (**84**) for a point source located at $(x_0,\,0)$ in the flat case read

$$c_0=-x_0\qquad c_2=-2x_0\qquad c_4=2x_0\qquad \ldots\ .$$

Recalling to the wave progress difference expansion (**87**), note that we are completely free to choose the position of the reference source, for the absolute term is always possible to match by phase constant(s) as $\delta=c-\tilde{c}$ from (**86**). Hence, we for the source position obtain

$$x_0=-{}^1\!/_2\ .$$

This restores the Gaussian focus from (**47**) as a place, from which apparently emerge the spherical waves of focused wavefront (see Fig. 3.).

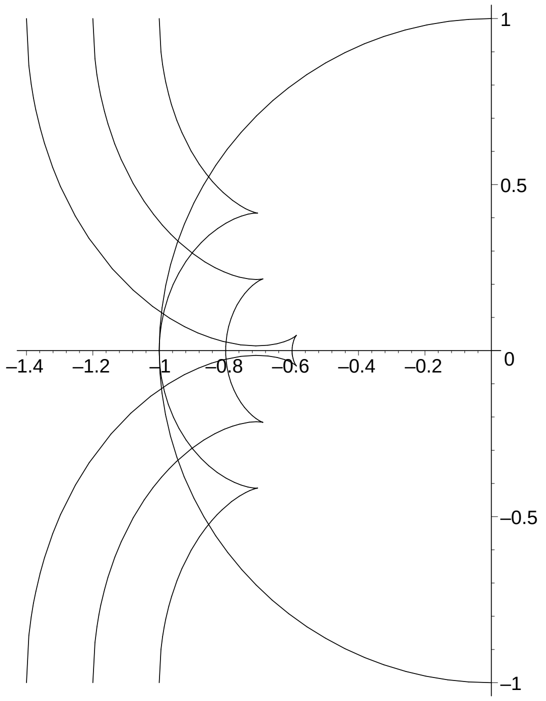



Fig. 3. *Several wavefronts of reflected rays. Note the way they are closing to the focus point. The parts of the wavefronts that are outside the mirror are unphysical – those belong to the rays, that would actually survey the second (and possibly even higher number of) reflections, which is however not implemented in the formulas hereby used.*

Also, (empirical) *eikonal along ray* gets

$$\psi_Y = \sqrt{1-Y^2} + \sqrt{(Y-y)^2 + (-\sqrt{1-Y^2}-x)^2},$$

whence using the ray equation (**42**) we obtain

$$\tau = \psi_Y - \sqrt{1-Y^2}\ .$$

and the wavefront (**46**) can be re-parameterised using this physical phase as

$$X_w = (1-2Y^2)(\psi_Y - \sqrt{1-Y^2}) - \sqrt{1-Y^2}$$
$$Y_w = -2Y\sqrt{1-Y^2}(\psi_Y - \sqrt{1-Y^2}) + Y\ .$$

## The Higher-Dimensional Optics

In this chapter we will study the higher-dimensional spacetimes possessing the maximal spherical symmetry. The basic equations will be provided; particularly, it will be shown that in all these spacetimes the equatorial section brings formally same results as in the commonly dimensional case, justifying the course taken in the rest of this work.

For the purpose of this application, we shall disregard the possibility of arbitrariness of ray coordinate differences direction (as found in (**18**) etc. ) and we also consider only the situation before turning points as they would bring no qualitative difference.

Recall the static $(d-2)$-dimensional spherically symmetric metric element (**6**)

$$\mathrm{d}s^2 = g_{\mathrm{t}}(r)\,\mathrm{d}t^2 - g_{\mathrm{r}}\,\mathrm{d}r^2 - r^2(\,\mathrm{d}\Omega_{d-2})^2,$$

where

$$(\,\mathrm{d}\Omega_{d-2})^2 = (\,\mathrm{d}\varphi^{d-2})^2 + \sin^2\varphi^{d-2}\Big[(\,\mathrm{d}\varphi^{d-3})^2 + \sin^2\varphi^{d-3}\big[(\,\mathrm{d}\varphi^{d-4})^2 + \ldots\big]\Big]\,.$$

The eikonal equation $\psi_{,i}\psi^{,i}{=}0$ is separable and the complete (impulse) integral reads

$$\psi - \psi_0 = \omega t - \int\sqrt{\omega^2\frac{g_{\mathrm{r}}}{g_{\mathrm{t}}} - \Psi_{d-2}^2\frac{g_{\mathrm{r}}}{r^2}}\,\mathrm{d}r - \sum_{n=1}^{d-2}\int\sqrt{\Psi_n^2 - \frac{\Psi_{n-1}^2}{\sin^2\varphi^n}}\,\mathrm{d}\varphi^n \qquad (\mathbf{49})$$

if set $\Psi_0{=}0$. The integrals

$$I_n \equiv \int\sqrt{\Psi_n^2 - \frac{\Psi_{n-1}^2}{\sin^2\varphi^n}}\,\mathrm{d}\varphi^n$$

are elementary, namely

$$I_n = -\frac{1}{2}\Psi_{n-1}\mathrm{arctan}\left(\frac{2\cos\varphi^n\Psi_{n-1}\sqrt{\Psi_n^2\sin^2\varphi^n - \Psi_{n-1}^2}}{\Psi_{n-1}^2\cos^2\varphi^n + \Psi_{n-1}^2 - \Psi_n^2\sin^2\varphi^n}\right) - \Psi_n\mathrm{arctan}\left(\frac{\Psi_n\cos\varphi^n}{\sqrt{\Psi_n^2\sin^2\varphi^n - \Psi_{n-1}^2}}\right)\,.$$

Note that $I_1{=}\Psi_1\varphi^1$. The (canonical) ray equation is given by a system

$$\frac{\partial}{\partial\Psi_{d-2}}:\qquad 0{=}\int\frac{\Psi_{d-2}\sqrt{g_{\mathrm{r}}}}{r^2\sqrt{\frac{\omega^2}{g_{\mathrm{t}}}-\frac{\Psi_{d-2}^2}{r^2}}}\,\mathrm{d}r - \int\frac{\Psi_{d-2}}{\sqrt{\Psi_{d-2}^2-\frac{\Psi_{d-3}^2}{\sin^2\varphi^{d-2}}}}\,\mathrm{d}\varphi^{d-2} \qquad (\mathbf{50})$$

$$\frac{\partial}{\partial\Psi_n}:\qquad 0{=}\int\frac{\Psi_n}{\sin^2\varphi^{n+1}\sqrt{\Psi_{n+1}^2-\frac{\Psi_n^2}{\sin^2\varphi^{n+1}}}}\,\mathrm{d}\varphi^{n+1} - \int\frac{\Psi_n}{\sqrt{\Psi_n^2-\frac{\Psi_{n-1}^2}{\sin^2\varphi^n}}}\,\mathrm{d}\varphi^n, \qquad n<d-2\,.$$

Particularly,

$$\frac{\partial}{\partial \Psi_1}: \qquad \mathbf{0} = \int \frac{\Psi_1}{\sin^2\varphi^2 \sqrt{\Psi_2^2 - \frac{\Psi_1^2}{\sin^2\varphi^2}}} \mathrm{d}\varphi^2 - \varphi^1 \ .$$

Note that

$$\int \frac{\Psi_n}{\sqrt{\Psi_n^2 - \frac{\Psi_{n-1}^2}{\sin^2\varphi^n}}} \mathrm{d}\varphi^n = -\arctan \frac{\Psi_n \cos\varphi^n}{\sqrt{\Psi_n^2 \sin^2\varphi^n - \Psi_{n-1}^2}}, \qquad \text{and}$$

$$\int \frac{\Psi_n}{\sin^2\varphi^{n+1}\sqrt{\Psi_{n+1}^2 - \frac{\Psi_n^2}{\sin^2\varphi^{n+1}}}} \mathrm{d}\varphi^{n+1} = -\frac{1}{2}\arctan\left(\frac{2\cos\varphi^{n+1}\Psi_n\sqrt{\Psi_{n+1}^2\sin^2\varphi^{n+1} - \Psi_n^2}}{\Psi_n^2\cos^2\varphi^{n+1} + \Psi_n^2 - \Psi_{n+1}^2\sin^2\varphi^{n+1}}\right) .$$

While the last member in fact equals the last term in eikonal, it can be plugged therein with vanishing of $\varphi^1$ therefrom; yielding for all the integrals involving $\varphi^2$ in the eikonal (**49**)

$$\int \frac{\Psi_2^2}{\sqrt{\Psi_2 - \frac{\Psi_1^2}{\sin^2\varphi^2}}} \mathrm{d}\varphi^2 \ .$$

Again, this integral can be (almost) substituted from the equation from $\partial/\partial\Psi_3$ and so on. After repeating this procedure $d-3$ times in all we finally get

$$\psi_\Psi = \omega t - \int \frac{\sqrt{g_\mathrm{r}}}{g_\mathrm{t}} \frac{\omega^2}{\sqrt{\frac{\omega^2}{g_\mathrm{t}} - \frac{\Psi_{d-2}^2}{r^2}}} \mathrm{d}r$$

(actually irrespective of dimension). The particular *eikonal along ray* (with time neglected)

$$\psi_\Psi - \psi_0 = -\int_{r_s}^{r} \frac{\sqrt{g_\mathrm{r}(y)}}{g_\mathrm{t}(y)} \frac{\omega^2}{\sqrt{\frac{\omega^2}{g_\mathrm{t}(y)} - \frac{\Psi_{d-2}^2}{y^2}}} \mathrm{d}y$$

may serve as determination of $r(\Psi_{d-2})$; the construction is as follows: let $\psi_\Psi - \psi_0$ be the wave distance from a source – a constant $\Phi$. Then we may write

$$f: \mathbf{0} = \Phi + \int_{r_s}^{r} \frac{\sqrt{g_\mathrm{r}(y)}}{g_\mathrm{t}(y)} \frac{\omega^2}{\sqrt{\frac{\omega^2}{g_\mathrm{t}(y)} - \frac{\Psi_{d-2}^2}{y^2}}} \mathrm{d}y \ .$$

The axial expansion

$$r(\Psi_{d-2}) \approx r|_{\Psi_{d-2}=0} + \left.\frac{\partial r}{\partial \Psi_{d-2}}\right|_{\Psi_{d-2}=0} \Psi_{d-2} + \frac{1}{2}\left.\frac{\partial^2 r}{\partial \Psi_{d-2}^2}\right|_{\Psi_{d-2}=0} {\Psi_{d-2}}^2 + \ldots$$

can be constructed from implicit derivatives. Indeed, while there are no singularities in the borders, with notation $r_0 \equiv r|_{\Psi_{d-2}=0}$ we can write

$$r_0: \qquad \mathbf{0} = f|_{\Psi_{d-2}=0} = \Phi + \int_{r_s}^{r_0} \sqrt{\frac{g_\mathrm{r}}{g_\mathrm{t}}}\,\omega\,\mathrm{d}y$$

$$\left.\frac{\partial r}{\partial \Psi_{d-2}}\right|_{\Psi_{d-2}=0} = -\left.\frac{f'_{\Psi_{d-2}}}{f'_r}\right|_{\Psi_{d-2}=0} = -\left.\frac{\displaystyle\int_{r_s}^{r} \frac{\sqrt{g_\mathrm{r}}}{g_\mathrm{t}} \frac{\omega \Psi_{d-2}}{y^2\sqrt{\frac{\omega^2}{g_\mathrm{t}} - \frac{\Psi_{d-2}^2}{y^2}}}\,\mathrm{d}y}{\displaystyle\frac{\sqrt{g_\mathrm{r}(r)}}{g_\mathrm{t}(r)} \frac{\omega^2}{\sqrt{\frac{\omega^2}{g_\mathrm{t}(r)} - \frac{\Psi_{d-2}^2}{r^2}}}}\right|_{\Psi_{d-2}=0} = 0$$

$$\left.\frac{\partial^2 r}{\partial {\Psi_{d-2}}^2}\right|_{\Psi_{d-2}=0} = -\frac{1}{\omega^2}\sqrt{\frac{g_\mathrm{t}(r_0)}{g_\mathrm{r}(r_0)}} \int_{r_s}^{r_0} \frac{\sqrt{g_\mathrm{r} g_\mathrm{t}}}{y^2}\,\mathrm{d}y$$

$$\vdots$$

if all limits used exist (primes mean differentiation according to subscript-stated quantity). In this way we have

$$r(\Psi_{d-2}) \approx r_0 - \frac{1}{\omega^2}\sqrt{\frac{g_\mathrm{t}(r_0)}{g_\mathrm{r}(r_0)}} \int_{r_s}^{r_0} \frac{\sqrt{g_\mathrm{r} g_\mathrm{t}}}{y^2}\,\mathrm{d}y\, {\Psi_{d-2}}^2 + \ldots$$

This expansion can be used in the topmost ray-equation component to obtain $\varphi^{d-2}(\Psi_{d-2}, \Psi_{d-3})$. Similarly, all other quantities, that are used in this work can be in the case of hereby considered configuration obtained, showing the more general validity of two-dimensional treatment.

## The Maxwell's Fish-Eye and Gravitational Lensing

This chapter deals with primary optical quantities within the framework of general relativity; namely, the spectacularity of *ideal optical instrument* in Maxwell's sense is put forward. It is shown that a new realisation of such a fundamental instrument can be achieved within general relativity by matter of suitable properties particular distribution choice, thus creating an interesting light deflector on – possibly – large extents; (some) consequences are discussed.

The grounds for relativistic optics are given by *testing* electromagnetic field presumption (i.e. field thanks to whose weakness the disturbance of the underlying spacetime is neglibigle) and its *geometrical optics* approximation in covariant form.

### The fish-eye – an ideal optical instrument

By definition, an *ideal optical instrument* images stigmatically (point-to-point) a three-dimensional domain. As a classical example, consider equatorial (plane) section of spherical symmetry adopted with polar coordinates as usual. Denoting $n$ intrinsic index of refraction, the ray equation

$$\varphi - \varphi_s = \int_{r_s}^{r} \frac{\varrho\, dy}{y\sqrt{y^2 n^2(y) - \varrho^2}} \tag{51}$$

has come from particular eikonal for a point source $[r_s, \varphi_s]$, where no turning points are considered; distinguishing the rays, parameter $\varrho$ forms together with $\varphi$ the *ray coordinates*.

Inspecting the square root from (**51**) in domain possessing (spherically symmetric) refractive index $n{=}n_m$, where

$$n_m \equiv \frac{n_0}{1 + (r/a)^2} \tag{52}$$

with $n_0$, $a$ being constant, one can easily conclude, that each ray is radially bound to region between the (square) roots found in which there lay turning points; the two roots (dependent of $\varrho$), however, need not lie in the same half-plane. Indeed, integration of (**51**) subject to (**52**) yields the rays

$$\begin{aligned}\varrho(r^2 - a^2) = \frac{ar}{ar_s}\Big[&\varrho({r_s}^2 - a^2)\cos(\varphi - \varphi_s) + \\ &+ \sqrt{a^4 {r_s}^2 n_0^2 - \varrho^2({r_s}^2 + a^2)^2}\,\sin(\varphi - \varphi_s)\Big],\end{aligned} \tag{53}$$

which is the equation of circles in polar coordinates that enclose the origin $\boldsymbol{r}\mathbf{=0}$ except case $\varrho$=0, when it describes a straight line passing through origin and source, see Fig. 1.

It can be noticed, that whenever $\sin(\varphi-\varphi_s)$ turns zero, (**53**) is fulfilled irrespective of ray parameter $\varrho$, i.e. all rays pass through the same and thus *focal point*. Hence, while the sine turns zero every $k\pi$, to every point source of light there exist an adjoint focus point $r_f$ which lies in opposite direction off the origin and the separation of these two foci as deduced from (**53**) is such that

$$r_s r_f = a^2 \, .$$

In this way, the smallest region occupied by 'fully working' fish-eye is thus a domain $\boldsymbol{r} \leq \boldsymbol{a}$ when the separation of the foci is minimal on the opposite poles of its border. A particular case of (**52**), being a ball of radius $R$ with central index $n_0$=2 and extent such that on the border sphere index of refraction is continuous towards air (vacuum), i.e. $a$=$R$ is often called the *Maxwell's fish-eye*; for historical treatment, see [**Bor**].

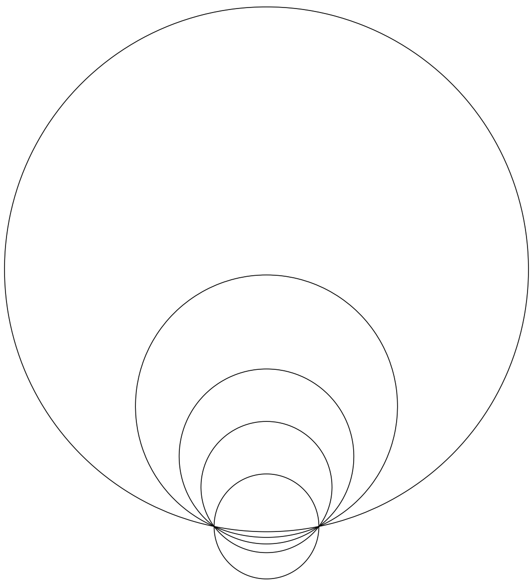


Fig. 1. *Several rays trajectories for a fish-eye according to (***53***). Smallest is the circle* $\boldsymbol{r}=\boldsymbol{a}$.

**Effective index of refraction within general relativity**

In scope of previous paragraphs we will study only static spherically symmetric spacetimes; although variety of these solutions is described generally by two undetermined coefficients $g_{\rm tt}, g_{\rm rr}$ in their metric which thus also govern the curvature of the spacetime, we confine ourselves in further to familiar case of 3+1 dimensional spacetime. The Fermat principle in such a configuration reads

$$\delta \int \frac{\mathrm{d}l}{\sqrt{g_{\rm tt}}} = 0,$$

which also shows, that in gravitational field considered the rays will never be shortest curves in space unless constant $g_{\rm tt}$ [**Lan**].

Compared with classical version of the principle, we could conclude that for *effective index of refraction* there stands

$$n_{\rm eff} = 1/\sqrt{g_{\rm tt}}\ . \tag{54}$$

Namely this approximation was used to construct the flowing media analogs [**LePi**] to relativistic light bending.

It is however not true that the rays would follow the same paths in the flat spacetime with intrinsic index of refraction of same form, because the exact ray equation in our (curved) case reads

$$\varphi = \int \frac{\varrho\sqrt{1 - 2m(r)/r}}{r\sqrt{r^2/g_{\rm tt} - \varrho^2}}\,\mathrm{d}r\ . \tag{55}$$

where $m(y)$ stands for mass enclosed within $r \leq y$. Upon comparison with (**51**) it is seen that there is an extra relativistic factor in the numerator of (**55**) which can generally not be retracted to the suitable position under the square root.

This disproportion actually shows that the index of refraction is post-classically unsuitable quantity to describe optics. Yet, we can say that the effective index of refraction (**54**) is true for week fields, where the relativistic factor gets approximated by one.

Let us for now keep the presumption of week fields, i.e. that the square root of metric coefficient coincides with the reciprocal value of effective index of refraction, and study two well-known static solutions in this limit. Such an approximation is appropriate, for it is expectable, that optically transparent media should be gravitationally weak.

For the exterior solution (of static black hole) with mass $M$ and charge $Q$, we from (**8**) get

$$n = \frac{1}{\sqrt{1 - 2M/r + Q^2/r^2}},$$

which unfortunately gives no acceptable approximation due to its nature.

The situation is far more optimistic for the case of interior solution realised by an orb or radius $r_0$ (as measured in critical radii of itself) constituted from perfect fluid; then, from (**7**) we have

$$n = \frac{1}{\frac{3}{2}\sqrt{1-\frac{1}{r_0}} - \frac{1}{2}\sqrt{1-\frac{r^2}{r_0^3}}},$$

even irrespective of (spatial) dimension. In the weak field as well as in near origin limits,

$$n \approx \frac{1}{\left(\frac{3}{2}\sqrt{1-\frac{1}{r_0}} - \frac{1}{2}\right) + \frac{r^2}{4r_0^3}},$$

driving such an object to behave like a fish-eye with parameters

$$n_0 = 1 \Big/ \left(\frac{3}{2}\sqrt{1-\frac{1}{r_0}} - \frac{1}{2}\right)$$
$$a^2 = 4r_0^3 \left(\frac{3}{2}\sqrt{1-\frac{1}{r_0}} - \frac{1}{2}\right).$$

According to previous we seek only configurations with $r_0 \geq a$, which brings the constraint

$$r_0 \in \left\langle \frac{9}{8}, \frac{5+\sqrt{27}}{8} \right\rangle,$$

where the lower bound comes from the stability condition $r_0 > {}^9\!/_8$ guarantying finite central value of pressure within the fluid. It is seen, that for the properties of fish-eye to manifest the cloud would have to be of nearly critical extent, which is at least contradiction to weak field limit, and, of course, the transparency of such an object is also out of question, as long as even for the Sun the photons travel from its interior several thousands years, while the number of their capture and re-emission lays beyond count.

It is time then to step to explicit solving Einstein equations subject to fish-eye creation conditions. Due to Birkhoff theorem [**Ste**] and spacetimes class chosen there are no more exterior solutions, whence we need to seek only within interior ones, as we indeed shell. We keep for the nearest part of the work the presumption of weak field, which allows us to make us of effective index of refraction simple definition (**54**).

**The (interior) solutions**

The behaviour of the quantities describing solution of Einstein's equations in the case chosen – the pressure $p(r)$, matter density $\rho(r)$, the gravitational filed potential $\Phi(r)$ and the enclosed mass-weight $m(r)$ – is governed by three equations [**Tho**] (note that $g_{\rm tt}={\rm e}^{2\Phi}$, $g_{\rm rr}=1/(1-2m/r)$) : the field-strength equation

$$\Phi' = \frac{m + 4\pi r^3 p}{r(r-2m)} \tag{56}$$

hydrostatic equilibrium equation

$$p' = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r-2m)} \tag{57}$$

and (defining) equation

$$m = 4\pi \int_0^r \xi^2 \rho(\xi) {\rm d}\xi \; . \tag{58}$$

In our case the situation is right contrary to usual one: we know the metric coefficient – deducing from (**54**) subject to (**52**) – and therefrom we try to find the matter that produces it – i.e. its distribution and properties (state equation). Following this view it is worth rewriting (**57**) to

$$p' = -(\rho + p)\Phi' \tag{59}$$

and as long region as matter distribution density is continuous and bound, (**58**) may be written as

$$m' = 4\pi r^2 \rho \; . \tag{60}$$

**a) the state equation for weak fields.** Considering $\Phi$ given, we are able to separate pressure from the field strength equation (**56**) and together with density from (**60**) it may be plugged into hydrostatic equilibrium equation (**59**); a linear first-order ODE for mass is obtained, whose solution is

$$m = \frac{r^3 {\rm e}^{\chi}}{(1 + r\Phi')^2} \left( {\rm Const} + \int (r\Phi'' + r\Phi'^2 - \Phi')(1 + r\Phi') \frac{{\rm e}^{\chi}}{r^2} \, {\rm d}r \right) \tag{61}$$

generally, with

$$\chi = 2\Phi - \int \frac{4\Phi'}{1 + r\Phi'} \, {\rm d}r \; .$$

To achieve the properties of the fish-eye, we need to write now $1/\sqrt{e^{2\Phi}}=n_m$ whence a direct choice

$$\Phi = \ln \frac{1+(r/a)^2}{n_0}$$

is obtained. Fixing integration constant in (**61**) such that $\rho|_{r=0}\equiv\rho_c$ we then have

$$m=\frac{4\pi a^{4/3}\rho_c}{3}\frac{r^3}{(3r^2+a^2)^{2/3}}$$
$$\rho=\frac{a^{4/3}\rho_c}{3}\frac{5r^2+3a^2}{(3r^2+a^2)^{5/3}}\quad(\mathbf{62})$$
$$p=\frac{1}{2\pi}\frac{(3r^2+a^2)^{2/3}-2\pi a^{4/3}\rho_c(5r^2+a^2)/3}{(a^2+r^2)(3r^2+a^2)^{2/3}}\ .$$

Note that along radial coordinate density is falling, mass is rising with $m_c$=0 and for pressure

$$p_c = \frac{1}{2\pi a^2} - \frac{\rho_c}{3},$$

hence, the class of solutions obtained seems physically reasonable. To keep central values of state quantities non-negative,

$$0 \le \rho_c \le 3/(2\pi a^2)$$

have to hold, keeping the density low in agreement with the weak-field presumption.

Within the quite broad class of solutions just presented, there is one physically very interesting. Namely, for the choice $\rho_c=3/(4\pi a^2)$ we get

$$p_c = \rho_c/3,$$

hence the relativistic fluid state equation in the centre of the body.

In correspondence with the reasoning from the first paragraph we must guarantee $R_0\geq a$. To do so, we use the freedom within relationship $a\sim\rho_c$: as can be seen, by choosing $\rho_c$ small enough, we can always acquire supercritical extent of physical cloud. Moreover it is in accordance with the approximation request to choose $\rho_c$ small.

We observe, that for sewing the metrics on the border we still have a degree of freedom in choosing of $n_0$ from the second equation. To do it, we first use the first one to exclude the one-third exponent, and gaining thus significant simplification, we can finally write

$$n_0 = \sqrt{1+5\frac{R_0^2}{a^2}}\ .$$

Our treatment is thus completely physical, as can be directly observed from the fact, that $n_0 > 1$ always holds.

In the following part of the work, we drop the effective index of refraction and will discuss more accurate approaches – the fact that the field should however not be strong (transparency) obviously holds.

**b) ad hoc mass.** The relativistic factor in (**55**) gets also approximed by one if

$$m(r) = \alpha r,$$

with α small. Such a choice does not restrict the freedom needed to construct the properties of the fish-eye, however, unphysical state quantities behaviour is obtained.

**c) the precise solution** Another way is to make such a choice that this relativistic factor particularly moved to the desired place within the square root. This happens for

$$m = \frac{r^2/g_{\mathrm{tt}} - n^2 r^2}{2\varrho/r - 2n^2 r} \, .$$

Note, however, that either mass or field strength would have to depend on ray parameter - behaviour evoking for a spacetime a vision of an anisotropic medium, which is completely physically unreasonable.

## The Focus of a Cluster and Its Aberrations

In this chapter, general idea of focusing is studied within the framework of optics extension into general relativity (covariant optics). In a configuration of static spacetime, the general, mathematically rigorous treatment is presented of rays, wavefronts and caustics of spherical symmetry, particularly with regard to problems of their obtaining within general relativity. As a concrete result, the cluster focal length and its aberration structure are obtained. In this way, a gravitational lensing situation is depicted in terms of being a real lens.

We will present the treatment in static, spherically symmetric case, allowing us – after choosing e.g. a point source of radiation – to unambiguously identify the *optical axis* with symmetry axis and make use of symmetry gained simplifications. From a mathematical insight [**Arn**], we then expect the caustic to be shaped as a revolution of cusp type catastrophe. Also, only spherical aberrations of however (generally) all orders are expected to rise for any axisymmetric source.

Let there be a static spherically symmetric solution to Einstein equations, valid in spacetime region $\Sigma$. In spherical coordinates $(r,\ \vartheta,\ \varphi)$, this generally admits the metric

$$\Sigma:\ \mathrm{d}s^2 = g_{\mathrm{t}}(r)\, c^2 \mathrm{d}t^2 - g_{\mathrm{r}}(r)\,\mathrm{d}r^2 - r^2(\,\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2), \tag{63}$$

with $c$ the speed of light. On a non-empty intersection $\sigma$ with equatorial (hyper)surface $\vartheta{=}\pi/2$ this brings

$$\Sigma|_{\vartheta=\pi/2} = \sigma:\ \mathrm{d}s^2 = g_{\mathrm{t}}\, c^2\mathrm{d}t^2 - g_{\mathrm{r}}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\varphi^2\ .$$

We will restrict ourselves to this cross-section.

#### The geometry

Let the solution of interest to Einstein equations consist of two metrics, properly sewed on $\boldsymbol{r{=}r_0}$ , with the point source of radiation at $[r_s,\ \varphi_s]$ lying not in the inner region: $r_s \geq r_0$.

The particular eikonal for a general ray passing into the inner region at $[r_0,\ \varphi_{\mathrm{in}}]$ and leaving it subsequently at $[r_0,\ \varphi_{\mathrm{out}}]$ after passing the turning point of $r{=}a$, as visualised by Fig. 1, gets

$$\begin{aligned}\theta\!:\psi-\psi_0=&\frac{\omega}{c}(t-t_0)-(-1)^k\frac{\omega}{c}\varrho(\varphi-\varphi_s)-\\&-(-1)^m\frac{\omega}{c}\int\limits_{r_s}^{r_0}-\int\limits_{r_0}^{r}\Theta_{\mathrm{outer}}(y)\mathrm{d}y-2(-1)^m\frac{\omega}{c}\int\limits_{r_0}^{a}\Theta_{\mathrm{inner}}(y)\mathrm{d}y,\end{aligned}$$

where

$$\Theta(y) \equiv \sqrt{g_{\mathrm{r}}(y)\left(\frac{1}{g_{\mathrm{t}}(y)} - \frac{\varrho^2}{y^2}\right)}$$

are integrands from (**18**), with the subscript choosing solution, whose metric coefficients are appropriate. Then particular ray equation gets

$$\pi : \varphi - \varphi_s = -(-1)^{k+m} \int_{r_s}^{r_0} - \int_{r_0}^{r} \Pi_{\mathrm{outer}}(y)\, \mathrm{d}y - 2(-1)^{k+m} \int_{r_0}^{a} \Pi_{\mathrm{inner}}(y),$$

where

$$\Pi(y) = \frac{\partial\Theta(y)}{\partial\varrho} \equiv \frac{\varrho\sqrt{g_{\mathrm{r}}(y)}}{y^2\sqrt{\frac{1}{g_{\mathrm{t}}(y)} - \frac{\varrho^2}{y^2}}}$$

are integrands from (**20**). In this way, only those situations, when rays, that enter the inner solution region, exhibit in it its (single) turning point and after leaving to outer one, they (from symmetry) show no other turning points, are taken into account.

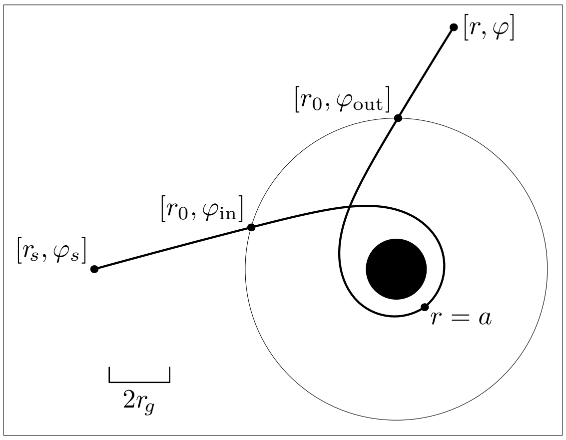



Fig. 1. *The sketch of geometrical situation concerning a ray (thick curve): within the great circle the inner solution is valid, with the black ring showing extent of mass critical radius. Note, that no further scaling information is needed, if radial coordinate is expressed in critical radii.*

Let us now formally evaluate the caustic $\kappa = \partial\pi/\partial\varrho$. As there are no turning points within outer solution, the integrands exhibit no singularities up to its border as well as the integration end-points are simply constant there. Thus,

$$\kappa : \int_{r_s}^{r_0} - \int_{r_0}^{r} \mathrm{K}_{\mathrm{outer}}(y)\, \mathrm{d}y + 2\frac{\partial}{\partial\varrho} \int_{r_0}^{a} \Pi_{\mathrm{inner}}(y)\, \mathrm{d}y = 0,$$

where

$$\mathrm{K}(y)=\frac{\partial\Pi(y)}{\partial\varrho}\equiv\frac{c}{\omega}\frac{\sqrt{g_{\mathrm{r}}(y)}}{g_{\mathrm{t}}(y)}\frac{1}{y^2\sqrt{\frac{1}{g_{\mathrm{t}}(y)}-\frac{\varrho^2}{y^2}}^3}$$

are integrands as in (**23**). The lengthy calculation according to (**22**) is not required in second term , if we happen to know the value $J$ of the full angular accrument along the ray in the fluid analytically:

$$2\int_{r_0}^{a}\Pi_{\mathrm{inner}}(y)\,\mathrm{d}y = J(\varrho);$$

in that case, we finally obtain the equation of caustic in form

$$\boldsymbol{\varphi-\varphi_s=-(-1)^{k+m}J-(-1)^{k+m}\left(\int_{r_s}^{r_0}-\int_{r_0}^{r}\right)\frac{\varrho\sqrt{g_{\mathrm{r}}}}{y^2\sqrt{\frac{1}{g_{\mathrm{t}}}-\frac{\varrho^2}{y^2}}}\mathrm{d}y}$$

$$\boldsymbol{0=\frac{\partial J}{\partial\varrho}+\left(\int_{r_s}^{r_0}-\int_{r_0}^{r}\right)\frac{\sqrt{g_{\mathrm{r}}}}{g_{\mathrm{t}}}\frac{\mathrm{d}y}{y^2\sqrt{\frac{1}{g_{\mathrm{t}}}-\frac{\varrho^2}{y^2}}^3}}$$

where all metric coefficients present are the outer solution ones. To obtain caustic in parametric form, the second of equations must be understood as implicit equation for $r(\varrho)$, and subsequently the first one as an explicit equation for $\varphi(\varrho,r(\varrho))$. Even though $\varrho$ is not generally the value of turning point radial coordinate, still, $\varrho=0$ is the only ray passing through origin. Thus, an expansion in the vicinity of optical axis (coming from symmetry) is acquired by expanding the coordinates for small $\varrho$. Using implicit derivatives formulas we obtain general expression of caustic

$$\begin{aligned}
r(\varrho)=r(0)+\frac{r(0)^2}{g_{\mathrm{t}}\sqrt{g_{\mathrm{r}}}}\cdot\left.\frac{\partial^2 J}{\partial\varrho^2}\right|_0\cdot\varrho+\frac{r^3(0)}{g_{\mathrm{t}}^3 g_{\mathrm{r}}}\Bigg[\left(\left.\frac{\partial^3 J}{\partial\varrho^3}\right|_0+3\left(\int_{r_s}^{r_0}-\int_{r_0}^{r(0)}\right)\frac{\sqrt{g_{\mathrm{r}}}g_{\mathrm{t}}^2}{y^4}\mathrm{d}y\right)\frac{\sqrt{g_{\mathrm{r}}}g_{\mathrm{t}}^2}{r(0)}-\\
-\frac{g_{\mathrm{t}}g_{\mathrm{r}}'r(0)+2r(0)g_{\mathrm{r}}g_{\mathrm{t}}'-4g_{\mathrm{r}}g_{\mathrm{t}}}{2g_{\mathrm{r}}}\left(\left.\frac{\partial^2 J}{\partial\varrho^2}\right|_0\right)^2\Bigg]\frac{\varrho^2}{2}+\ldots\\
\varphi(\varrho)=\left(\varphi_s-(-1)^{k+m}J(0)\right)-(-1)^{k+m}\left(\left.\frac{\partial J}{\partial\varrho}\right|_0+\left(\int_{r_s}^{r_0}-\int_{r_0}^{r(0)}\right)\frac{\sqrt{g_{\mathrm{t}}g_{\mathrm{r}}}}{y^2}\mathrm{d}y\right)\varrho-\\
-(-1)^{k+m}\left(1+\frac{2}{\sqrt{g_{\mathrm{t}}}}\right)\cdot\left.\frac{\partial^2 J}{\partial\varrho^2}\right|_0\cdot\frac{\varrho^2}{2}+\ldots
\end{aligned}\qquad(\mathbf{64})$$

with $r(0) \equiv r|_{\varrho=0}$ defined implicitly as

$$r(0): \int_{r_s}^{r_0} - \int_{r_0}^{r} \frac{g_t\sqrt{g_r}}{y^2}\,dy + \left.\frac{\partial J}{\partial \varrho}\right|_0 = 0;$$

outside integrals, all metric coefficients in last equations are to be treated as evaluated in $r(0)$.

The equation of projection of eikonal along ray (**21**) gets

$$\boldsymbol{\lambda: \psi_\varrho - \psi_0 = -(-1)^m \frac{\omega}{c} \int_{r_s}^{r_0} - \int_{r_0}^{r} \Lambda_{\text{outer}}(y)\,dy - 2(-1)^m \frac{\omega}{c} \int_{r_0}^{a} \Lambda_{\text{inner}}(y)\,dy,}$$

where

$$\Lambda(y) = \Theta(y) + \varrho \frac{\partial \Theta(y)}{\partial \varrho} \equiv \frac{\sqrt{g_r}}{g_t\sqrt{\dfrac{1}{g_t} - \dfrac{\varrho^2}{y^2}}}$$

are integrands from (**21**). Taking now the equation $\psi_\varrho$**=const** of constant phase accrument along ray for implicit expression for $r(\varrho)$ of the wavefront (into the constant, the signs and factor $\omega/c$ are set to stick to geometrical substantiality of wavefront) and using ray equation similarly to the case of caustic, one finally obtains the parametric expression of wavefront in form

$$\begin{aligned}
r(\varrho) &= r(0) + \sqrt{\frac{g_t}{g_r}} \cdot \left.\frac{\partial I}{\partial \varrho}\right|_0 \cdot \varrho + \\
&\quad + \sqrt{\frac{g_t}{g_r}} \left[ \left.\frac{\partial^2 I}{\partial \varrho^2}\right|_0 + \int_{r_s}^{r_0} - \int_{r_0}^{r(0)} \frac{\sqrt{g_r g_t}}{y^2}\,dy - \frac{1}{2}\left(\left.\frac{\partial I}{\partial \varrho}\right|_0\right)^2 \frac{g_r' g_t - g_r g_t'}{\sqrt{g_r}^3 \sqrt{g_t}} \right] \frac{\varrho^2}{2} + \ldots \\
\varphi(\varrho) &= \left(\varphi_s - (-1)^{k+m} J(0)\right) - (-1)^{k+m} \left( \left.\frac{\partial J}{\partial \varrho}\right|_0 + \int_{r_s}^{r_0} - \int_{r_0}^{r(0)} \frac{\sqrt{g_t g_r}}{y^3}\,dy \right) \varrho - \\
&\quad - (-1)^{k+m} \left(1 - 2\frac{g_t}{r(0)^2} \cdot \left.\frac{\partial I}{\partial \varrho}\right|_0\right) \cdot \left.\frac{\partial^2 J}{\partial \varrho^2}\right|_0 \cdot \frac{\varrho^2}{2} + \ldots
\end{aligned} \qquad (\mathbf{65})$$

with $r(0)$ defined implicitly now from

$$-\int_{r_s}^{r_0} - \int_{r_0}^{r(0)} \sqrt{\frac{g_r}{g_t}}\,dy - I(0) = \text{const};$$

$I(\varrho)$ is the ray total phase accrument along ray within the inner part of solution,

$$2\int_{r_0}^{a} \Lambda_{\text{inner}}(y)\,\mathrm{d}y = I(\varrho)\ .$$

**Particular metric and the results**

Let us choose for particular configuration the inner constant mass-density fluid solution and the outer, Schwarzschild one [**Ste**], with the field of electro-magnetic point source testing. On an equatorial section $\vartheta{=}{}^{\pi}/_{2}$ we obtain

$$\mathrm{d}s^2{=}\left[\frac{3}{2}\sqrt{1-\frac{r_g}{r_0}}-\frac{1}{2}\sqrt{1-\frac{r_g r^2}{{r_0}^3}}\right]^2 c^2\mathrm{d}t^2-\frac{1}{1-\dfrac{r_g r^2}{{r_0}^3}}\,\mathrm{d}r^2-r^2\mathrm{d}\varphi^2, \qquad r\le r_0$$

$$\mathrm{d}s^2{=}\left(1-\frac{r_g}{r}\right)c^2\mathrm{d}t^2-\frac{1}{1-\dfrac{r_g}{r}}\,\mathrm{d}r^2-r^2\mathrm{d}\varphi^2, \qquad r\ge r_0\ .$$

with $r_g$ the critical radius of matter involved, $r_g{=}(2MG/c^2)$ in SI units.

Within the Schwarzschild solution, all integrals involved in (**18**)—(**22**) are elliptic, approving the introduction. However, owing to the advantage of focus definition, the reduced integrals within expansion coefficients of (**64**), (**65**) are elementary. Recalling that $\varrho\ge 0$ was chosen, we state the turning points for metrics chosen,

$$a_{\text{Schw}} = \frac{2\varrho}{\sqrt{3}}\cos\left(\frac{\pi}{3}-\frac{1}{3}\arccos\frac{3\sqrt{3}}{2\varrho}\right) \qquad a_{\text{fl}} = \frac{3\frac{r_0}{\varrho}\sqrt{1-\frac{r_g}{r_0}}-\sqrt{\frac{{r_0}^2}{\varrho^2}-\frac{2r_g}{r_0}+\frac{9r_g^2}{4{r_0}^2}}}{2\frac{r_0}{\varrho^2}+\frac{r_g}{2{r_0}^2}}\ .$$

It is a matter of lengthy discussion of technical kind, that their behaviour is as expected in previous section, i.e. for rays closing to origin from high radial values, the stated expressions are the only turning points present; moreover, for all rays that are to enter fluid, $a_{\text{Schw}} < r_0$ holds. More interestingly, for a fluid chosen,

$$J = -\pi - \sum_{\pm}\arcsin\frac{3\varrho^2 r_g(r_0-r_g)/\sqrt{r_0}\mp\varrho^2 r_g\sqrt{r_0-r_g}-2{r_0}^3\left(\sqrt{r_0}\mp\sqrt{r_0-r_g}\right)}{r_0\left(\sqrt{r_0-r_g}\mp\sqrt{r_0}\right)\sqrt{4{r_0}^4-8\varrho^2 r_g r_0+9\varrho^2 r_g^2}},$$

where the double signs stand for summing two terms within $J$, once with upper signs and once with the lower ones. Then $r|_{\varrho=0}$ for caustic becomes

$$r(0) = \frac{{r_0}^2 r_s}{(3r_g r_s - {r_0}^2)}, \tag{66}$$

which is the radial coordinate value of caustic axial point, i.e. the focus of the lens position. Note that though emerged from expansions, by definition of the focus, this value is exact. Following the calculations, we can for the metrics chosen write

$$I = -4\frac{\omega {r_0}^2}{c r_g\sqrt{8\frac{r_0}{r_g}-9}}\left(\frac{\pi}{2}+\arcsin\frac{\sqrt{r_0}^3(3r_g-2r_0)}{\sqrt{r_0-r_g}\sqrt{4{r_0}^4+9\varrho^2 r_g^2-8\varrho^2 r_g r_0}}\right).$$

Adopting now the general form of axial wavefront $\boldsymbol{h}$**=const** within Schwarzschild geometry (**83**) is

$$\begin{aligned} h=&[r+r_g\ln(r-r_g)+c_0]+\\ &+\frac{r}{-2+rc_2}(\varphi-\varphi_A)^2+\frac{-\frac{1}{2}r_g+\frac{2}{3}r+r^4c_4}{(-2+rc_2)^4}(\varphi-\varphi_A)^4+\ldots, \end{aligned} \qquad (\mathbf{67})$$

i.e. to completely describe such wavefront, a single constant in every order of expansion is to be specified. To find the value of $\varphi_A$ around which to expand the wavefront, we proceed as follows. The choice of point source has unambiguously given rise to optical axis as coordinate line passing through source and origin. The optical axis is thus realized by ray $\varrho$=0 which gives e.g. from (**65**)

$$\varphi=\varphi_s-(-1)^{k+m}J(0)=\varphi_s+(-1)^{k+m}\pi,$$

which indeed is the continuation of coordinate line $\boldsymbol{\varphi=\varphi_s}$. Note, that the same holds for caustic (**64**), i.e. the cusp of a caustic, which is also the focus point, lies on this axis, as anticipated in introduction. The integers $k$, $m$ also confirm to (but) rule the orientation of the ray(s).

Substituting into general expression (**67**) the equation (**65**) for eikonal along ray and re-expanding in powers of $\varrho$ enables us to find the values of aberration constants in form

$$\begin{aligned} c_0=&-I(0)-2[r_0+r_g\ln(r_0-r_g)]+r_s+r_g\ln(r_s-r_g)\\ c_2=&2\frac{3r_sr_g-{r_0}^2}{{r_0}^2r_s}\\ c_4=&-\frac{1}{3{r_0}^6}(81r_g^3-108r_g^2r_0+15r_g{r_0}^2+16{r_0}^3)+\frac{2}{3{r_s}^3}-\frac{1}{2}\frac{r_g}{{r_s}^4}\\ &\vdots \end{aligned} \qquad (\mathbf{68})$$

To find the wave aberration in terms of wave progress difference, let us state the aberration coefficients (**85**) within Schwarzschild solution for axial point source $[r_s', \varphi_A]$ wavefronts

***h′*=const′**before turning point:

$$c_0' = r_s' + r_g \ln(r_s' - r_g)$$
$$c_2' = \frac{2}{r_s'}$$
$$c_4' = \frac{1}{6}\frac{3r_g - 4r_s'}{r_s'^4}$$
$$\vdots$$

Now, as a basis of aberration formulation, the first two terms in expansion of wave progress difference (**87**),

$$0 = (c_0 - c_0' - \text{const} + \text{const}') + \frac{r^2(c_2' - c_2)}{(-2 + rc_2)(-2 + rc_2')}(\varphi - \varphi_A)^2 + \\ + \left[\frac{-\frac{r_g}{2} + \frac{2r}{3} + r^4 c_4}{(-2 + rc_2)^4} - \frac{-\frac{r_g}{2} + \frac{2r}{3} + r^4 c_4'}{(-2 + rc_2')^4}\right](\varphi - \varphi_A)^4 + \ldots$$

can be always annihilated by suitable choice of reference (point) source position and phase. Namely here, for any $r_s'$, the const$'$ can be set to equal the absolute term within last equation zero, and, confronting the second aberration coefficients, we obtain

$$c_2 = c_2' : \qquad r_s' = \frac{{r_0}^2 r_s}{3 r_s r_g - {r_0}^2}$$

in direct agreement with (**66**). In this way, within Gaussian optics, the focus point is the apparent point source for emerging wavefronts. The first non-zero term gives rise to the wave progress difference expansion

$$h - h' = \frac{r^4(c_4 - c_4')}{(-2 + rc_2)^4}(\varphi - \varphi_A)^4 + \ldots \qquad \textbf{(69)}$$

which is the lowest spherical aberration term. The behaviour of higher order terms is similar: the condition that the wave progress difference is zero only if the wavefronts are identical ($c_k = c_k'$) is manifest; note however, that for such behaviour, the general potentiality of lowest two aberration coefficients identification is crucial. This behaviour forms the fundamental of Gaussian optics.

**Conclusion**

In this chapter, a general way of mathematically rigorous manipulation with optical ideas of focusing via caustic study within the frame of the general relativity has been presented. Expressions for caustic (**64**) and wavefront in the sense of the eikonal along ray (**65**) for testing electro-magnetic field on the equatorial section of static spherically symmetric space-time were obtained. As a consequence, upon choosing particular configuration, the exact value of perfect fluid lens focus (**66**) was given for a testing-field point source,

$$r_f \approx \frac{{r_0}^2}{3r_g},$$

here in far source limit.

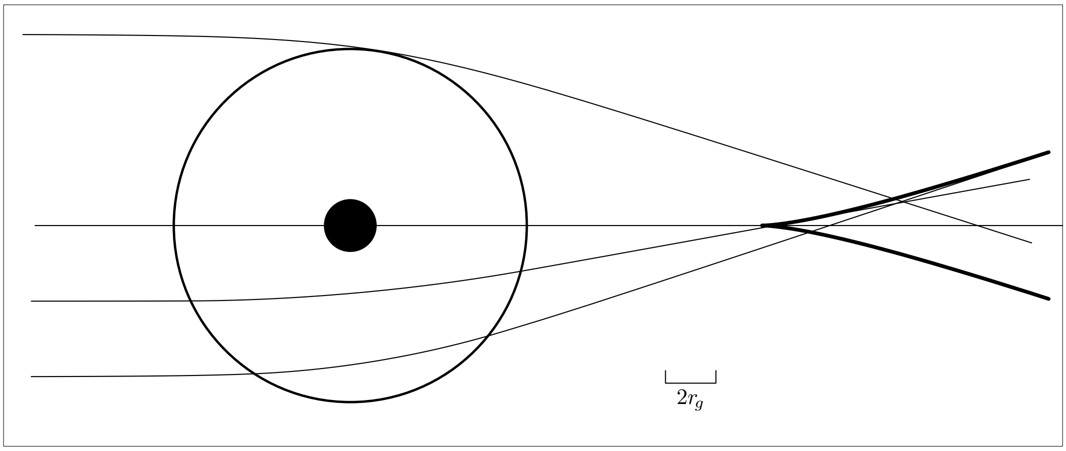



Fig. 2. *The caustic (thick curve) situation of cluster with $r_0$=$7r_g$ for far source configuration. The caustic cusp point (the focus of a system) is then at $r$=$16^1/_3 r_g$. Several inner rays are shown, all inevitably touching the caustic, that actually extends to radial infinity before touching the boundary ray. Hence, despite the diffractional corrections – that make the intensity along the caustic appropriately smaller especially for low-dense matter – the optical influence of cluster intervenes significant range of ambient universe*

Also, the constants (**68**) in the aberration expansion of the wavefront were obtained, moreover, using momentum eikonal formulas only. In addition, a comparison with point source aberrations was performed, confirming the Gaussian position of focus. The expansion of wave progress difference (**69**) was acquired, which starts with lowest spherical aberration term, as expected. The reason, why only the axial expansion is needed, lies in the fact, that the optical influence for the cases of interest is inconsiderable only for good alignment of source, deflector and observer.

Making use of the general treatment presented in this work, different configurations results are easily obtainable.

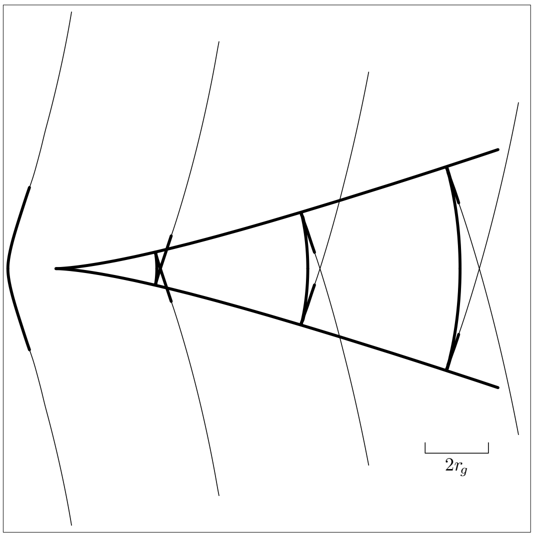



Fig. 3. *The situation for far source wavefronts near detail of caustic (thick curve). Depicted are the phase equip-spaced wavefronts, the bold segments showing the inner region traversed part. The $\varrho$ parameter extent is same for all wavefronts shown. As can be noted, the caustic indeed serves as a set of wavefronts singularities. Also note, that the orientation of caustic is opposite to case of reflection on a spherical mirror. In other words, the spherical aberration (of lowest order at least) for the cluster is of opposite sign to mirror one.*

## The Light within the FLRW Models

The general metric element (**5**), obeying the requirements of the cosmological principle (that can be formulated as homogeneity and isotropy of the Universe) reads

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)\,\mathrm{d}l^2,$$

where with the foliation presented, $\mathrm{d}l^2$ has the meaning of space arc-length element, whereas

$$\mathrm{d}l^2 = \mathrm{d}w^2 + f_k^2(w)(\,\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2),$$

with

$$f_k(w) = \begin{cases} \dfrac{1}{\sqrt{k}}\sin(\sqrt{k}w) & k>0,\ w\in\langle 0,\pi\rangle \\ w & k=0,\ w\in\langle 0,1\rangle \\ \dfrac{1}{\sqrt{-k}}\sinh(\sqrt{-k}w) & k<0,\ w\in\langle 0,\infty\rangle\ . \end{cases}$$

As is well known [**Bur**], the three cases stated cover the global topology of the space section of a 3-sphere, $R^3$, and 3-pseudo-sphere, respectively. In physical terms they are the closed, the flat and the open universes.

A complete impulse integral to eikonal equation $\psi^{,k}\psi_{,k}=0$ in our case is now

$$\psi = \int \frac{\lambda\,\mathrm{d}t}{a(t)} - (-1)^k\Psi\varphi - (-1)^l \int \sqrt{C - \frac{\Psi^2}{\sin^2\vartheta}}\,\mathrm{d}\vartheta - (-1)^m \int \sqrt{\lambda^2 - \frac{C}{f_k^2(w)}}\,\mathrm{d}w\ .$$

On the equatorial section $\vartheta={}^{\pi}/_{2}$ (with the auxiliary relation $C\equiv\Psi^2$) this reads

$$\psi = \int \frac{\lambda\,\mathrm{d}t}{a} - (-1)^k\Psi\varphi - (-1)^m \int \sqrt{\lambda^2 - \frac{\Psi^2}{f_k^2}}\,\mathrm{d}w,$$

whence the ray equation

$$\boldsymbol{\varphi = (-1)^{k+m} \int \frac{\varrho/f_k}{\sqrt{f_k^2 - \varrho^2}}\,\mathrm{d}w}, \tag{70}$$

with the usual notation $\Psi=\lambda\varrho$ towards the multiplicative constant. The appropriate eikonal along ray gets

$$\frac{\psi_\varrho}{\lambda} = -(-1)^m \int \frac{f_k\,\mathrm{d}w}{\sqrt{f_k^2 - \varrho^2}}\ .$$

The rays behaviour can be qualitatively found from the rooted expression decomposition

$$(f_k(w) + \varrho)(f_k(w) - \varrho),$$

whose vanishing can be understood graphically as crossing of horizontal lines $\pm\varrho$ with three curves: sin, id, sinh in the correct intervals (see Fig. 1.).

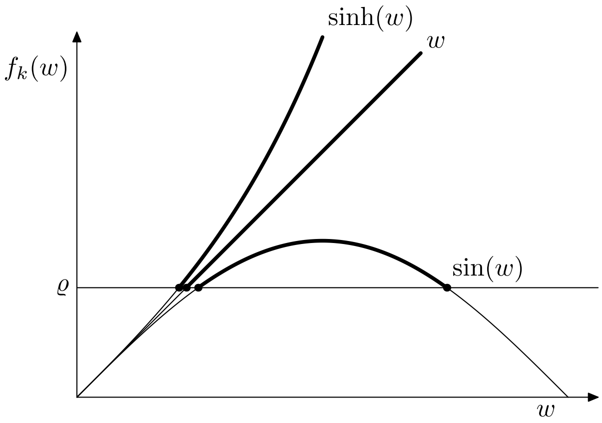



Fig. 1. *The rays behaviour in the three FLRW cases. The rays can exist only within intervals of angular values, where the curves are bold*

The negative $\varrho$ can be omitted without loss generality, and thus we can conclude, that in the flat case, the rays minimal radial coordinate is $\varrho$ as expected. The behaviour of rays in open case is qualitatively the same, but for the closed one, the value of $w$ oscillates between two values. As for the open case, the reason for such behaviour resides in monotony of $f_k(w)$; the closed solution ray will thanks to symmetry of $f_k(w)$ exhibit two turning points, which lie symmetrically with respect to the equator. The particular ray equation is now of a form

$$\varphi - \varphi_0 = (-1)^{k+m} \int_{w_0}^{f_k^{\text{inv}}(\varrho)} \pm \int_{f_k^{\text{inv}}(\varrho)}^{w} \frac{\varrho / f_k(\chi)}{\sqrt{f_k^2(\chi) - \varrho^2}} \, d\chi,$$

where $\chi = f_k^{\text{inv}}(\varrho)$ is the solution of equation $f_k(\chi) = \varrho$, i.e. (in the same order),

$$f_k^{\text{inv}}(\varrho) = \begin{cases} \dfrac{1}{\sqrt{k}} \arcsin(\sqrt{k}\,\varrho) \\ \varrho \\ \dfrac{1}{\sqrt{-k}} \operatorname{argsinh}(\sqrt{-k}\,\varrho)\ . \end{cases}$$

In closed case the expression should actually be a bit more complicated by a sum of full periods finished between the turning points. Note, however, that for the finally collapsing universe there is not enough time for the light to perfect more than one full period.

Using the procedure from the chapter ***Parametric derivatives of singularity containing***

***integrals*** we obtain a caustic in form

$$0=\int_{w_0}^{f_k^{inv}(\varrho)}\frac{-f_k^3(\chi)(f_k^{inv}(\varrho)-w_0)+\varrho f_k'(\chi)(\chi-w_0)f_k^{inv\prime}(\varrho)(2f_k^2(\chi)-\varrho^2)}{f_k^2(\chi)(f_k^2(\chi)-\varrho^2)^{3/2}(f_k^{inv}(\varrho)-w_0)}\mathrm{d}\chi\pm$$
$$\pm\int_{f_k^{inv}(\varrho)}^{w}\frac{-f_k^3(\chi)(f_k^{inv}(\varrho)-w)+\varrho f_k'(\chi)(\chi-w)f_k^{inv\prime}(\varrho)(2f_k^2(\chi)-\varrho^2)}{f_k^2(\chi)(f_k^2(\chi)-\varrho^2)^{3/2}(f_k^{inv}(\varrho)-w)}\mathrm{d}\chi\ .$$

Owing to the simplicity of the configuration chosen, we can however state the caustic also from the particular ray equation (coming from (**70**))

$$\varphi-\varphi_0=\begin{cases}\left[\frac{\pi}{2}+\arctan\frac{\varrho\cos w_0}{\sqrt{\sin^2 w_0-\varrho^2}}\right]\pm\left[\frac{\pi}{2}-\arctan\frac{\varrho\cos w}{\sqrt{\sin^2 w-\varrho^2}}\right]\\ \left[\frac{\pi}{2}+\arctan\frac{\varrho}{\sqrt{w_0^2-\varrho^2}}\right]\pm\left[\frac{\pi}{2}-\arctan\frac{\varrho}{\sqrt{w^2-\varrho^2}}\right]\\ \left[\frac{\pi}{2}+\frac{1}{2}\arctan\left(\frac{1}{2}\frac{2\varrho^2+\varrho^2\sinh^2 w_0-\sinh^2 w_0}{\varrho\cosh w_0\sqrt{\sinh^2 w_0-\varrho^2}}\right)\right]\pm\left[\frac{\pi}{2}-\frac{1}{2}\arctan\left(\frac{1}{2}\frac{2\varrho^2+\varrho^2\sinh^2 w-\sinh^2 w}{\varrho\cosh w\sqrt{\sinh^2 w-\varrho^2}}\right)\right],\end{cases}$$

where the pairing into the brackets visualises the terms from the individual integrals in particular ray equation (the right-angles always appear in a turning point). Thus the caustic gets

$$0=\begin{cases}\cos w_0\sqrt{\sin^2 w_0-\varrho^2}\mp\cos w\sqrt{\sin^2 w-\varrho^2}\\ \sqrt{w_0^2-\varrho^2}\mp\sqrt{w^2-\varrho^2}\\ \cosh w_0\sqrt{\sinh^2 w_0-\varrho^2}\mp\cosh w\sqrt{\sinh^2 w-\varrho^2}\ .\end{cases}$$

Using the non-equivalent modification we would get the solutions $\cos w_0=\cos w$, $w_0=w$, $\cosh w_0=\cosh w$, respectively. With their check back in original equations the situation for the last two cases reduces to $w=w_0$ before a turning point a nothing after a turning point – i.e. in the flat case as well as in the open case there is the only caustic point: the source itself, the rays defocus inevitably. In the closed case both opposite solutions are valid, hence except the source there is one more caustic point behind the turning point: the associated focus point at the pole opposite to the source position.

Using the generalisations of the FLRW models, allowing for small inhomogeneities the caustic structure would get more complicated, however fully solvable by the constructions hereby presented.

# Part Two
# The Aberration Formulation

In this section a different, more abstract approach to relativistic optics formulation is introduced. Starting from the general existence of semi-geodesic coordinates, the optical system properties are reduced to complete description by a set of constants – the expansion ones. Afterwards, a matching with reference wavefront is performed, which gives a new set of constants, the aberration ones. It is shown, how in this way the Gaussian optics has been constructed within curved spacetimes.

## The Optical Axis

Recall a general $(n+1)$-dimensional spacetime metric

$$\mathrm{g} = g_{\mathrm{tt}}(\mathrm{dt})^2 + 2g_{\mathrm{t}a}\,\mathrm{dt}\,\mathrm{d}x^a + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b, \qquad a, b = 1,\ 2,\ \ldots n \tag{71}$$

and suppose existence of time-separable coordinate eikonal $\psi$ with multiplicative constant $\omega$, i.e.

$$\psi = \omega\chi(t) - \omega h(x^a)\,\Big|\,\mathrm{g}(\mathrm{d}\psi,\ \mathrm{d}\psi) = 0, \tag{72}$$

which, of course, constraints g. The spatial projections of $\psi$**=const** form the wavefronts, which due to (**72**) are thus $\boldsymbol{h}$**=const**.

Let there henceforward be a space (true) length element $(\mathrm{d}l)^2$ given; following [**Lan**] the corresponding metric l comes from (arbitrary) metric (**71**) as

$$\mathrm{l} \equiv \tilde{\gamma}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = \left(\mp g_{ab} \pm \frac{g_{\mathrm{t}a}\,g_{\mathrm{t}b}}{g_{\mathrm{tt}}}\right)\mathrm{d}x^a\,\mathrm{d}x^b, \qquad \tilde{\gamma}^{ab} = \mp g^{ab} \tag{73}$$

provided signature $(\pm,\ \mp,\ \mp,\ \ldots)$ is used. As a Riemannean manifold can always be adopted with *semi-geodetic (normal) coordinates* $(x^{\mathrm{n}},\ x^{\alpha})$ [**Kor**], element (**73**) may be transformed to

$$\mathrm{l} = g_{\mathrm{nn}}(\mathrm{d}x^{\mathrm{n}})^2 + \gamma_{\alpha\beta}\,\mathrm{d}x^{\alpha}\,\mathrm{d}x^{\beta}, \qquad \alpha,\ \beta = 1,\ 2,\ \ldots n-1 \tag{74}$$

with $g_{\mathrm{nn}} > 0$ otherwise arbitrarily customisable. The meaning of $x^{\mathrm{n}}$ coordinate is such, that lines $\boldsymbol{x}^{\alpha}$**=const** are geodesics everywhere (locally) perpendicular to transversal hypersurfaces $T$: $\boldsymbol{x}^{\mathbf{n}}$**=const** whose metric $\tau$ is the transversal part of (**74**), i.e. $T$: $\tau{=}\gamma_{\alpha\beta}\,\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}$. As a matter of fact, there is still one degree of freedom, that allows for $g_{\mathrm{nn}}{=}1$, which makes $\Delta x^{\mathrm{n}}$ directly an arc length. We will however use the normal geodesic coordinates in the general form stated above.

An infinitesimal transversal element is then

$$\Delta l_{\tau} = \sqrt{\gamma_{\alpha\beta}\Delta x^{\alpha}\Delta x^{\beta}}\ .$$

It is just this infinitesimal arc, where there is no need to actually integrate along the geodesics connecting the points whose distance is seeked. The physical reason why we can use the infinitesimal transversal arc resides in the fact, that we aim to construct an aberrational formulation, which is to bring only expansions in powers of small values of the arcs. Thus, the infinitesimal arc is of high interest to optics, for the simplification it brings is crucial.

By expansion (of the wavefront) in the powers of transversal element we mean expansion

$$h = \sum_{i=0}^{\infty} h_i(x^{\mathrm{n}})\phi_i(\Delta l_{\tau}), \tag{75}$$

where, obviously, carrying out the expansion in $x^\kappa$ coordinates yields the full (Taylor) expansion of the wavefront.

We can now (with regard to Minkowski case compatibility) define an *optical axis*:

**By an optical axis we mean a coordinate curve $x^\kappa$=konst$^\kappa$, for which it holds, that the wavefronts of every point source lying ibidem are in its vicinity expanded in the powers of transversal arc only.**

Here comes the great utility of semi-geodesic coordinates within optics: as these coordinates exist on every Riemannean manifold, they can be made a general tool for study the optical properties. In every particular case, any of $x^\alpha$**=const** serves as optical axis candidate.

Note, that the above definition of optical axis is existence-like. It says, that we can pick a candidate for the axis and try whether the wavefront that is axial with respect to this axis allows for the expansion in transversal arc. It is only when it does, that the candidate was chosen correctly. Hence, it cannot be told in advance what will be the optical axis in a given spacetime (if any at all). This is the cost for allowing the curved spacetimes.

The general axial wavefront expansion is then

$$h = \sum_{k=0}^{\infty} h_k(x^n)\phi_k(\gamma_{\alpha\beta}\Delta x^\alpha \Delta x^\beta), \tag{76}$$

where the $\phi_k$ are the power-law functions. The exponent we must generally assume, will be a rational one – in this way, a square root of the arc could be moved into $\phi_k$; we expect the non-vanishing powers present in the last expansion to be of even order, i.e. of form $2(k-1)/(2n-1)$ with $k,\ n\in\mathbf{N}$, $n$ is fixed.

## The Axial Wavefront Expansion Coefficients

We shall consider in the following a static spacetime

$$\mathrm{d}s^2 = g_{\mathrm{tt}}\,\mathrm{d}t^2 - \mathrm{d}l^2,$$

where $\mathrm{d}l^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$, with $g_{\alpha\beta}(x^\kappa)$. As can be seen, such a spacetime can be foliated, and moreover, by true space sections. On these (Riemannean) sections, the *normal geodesic coordinates*

$$\mathrm{d}l^2 = g_{\mathrm{nn}}(\mathrm{d}x^{\mathrm{n}})^2 + \gamma_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$$

can always be introduced (at least locally).

For the wavefront ***h*=const** from (**76**) we of course require, that the corresponding coordinate eikonal $\psi = \omega t - \omega h$ fulfilled the eikonal equation, i.e.

$$g^{\mathrm{nn}} h_{,\mathrm{n}}^2 + \gamma^{\alpha\beta} h_{,\alpha} h_{,\beta} = g^{\mathrm{tt}}$$

We will now restrict our attention to our usual two-dimensional case, making profit from the fact, that the semi-geodesic metric (**74**) guarantees us within the two-dimensional space the possibility of introducing the orthogonal coordinates. In case of canonical sections, we can then have

$$\mathrm{d}l^2 = g_{\mathrm{nn}}(x^{\mathrm{n}})(\mathrm{d}x^{\mathrm{n}})^2 + g_{\mathrm{jj}}|_{x^{\mathrm{i}}=\mathrm{konst}}(\mathrm{d}x^{\mathrm{j}})^2$$

The transversal part (with respect to coordinate $x^{\mathrm{n}}$) of the arc is thus $g_{\mathrm{jj}}(\mathrm{d}x^{\mathrm{j}})^2$.

Let us now work out the definition of axial wavefront: the wavefront ***h*=konst** is guaranteed in separated form, and hence, its expansion in transversal part of arc can be stated as

$$h = \sum_{k=0}^{\infty} h_k(x^{\mathrm{n}})\phi_k(g_{\mathrm{jj}}(\Delta x^{\mathrm{j}})^2)\ .$$

The full expansion in powers of coordinate would be yielded by expanding the coefficient $g_{\mathrm{jj}}$.

The eikonal equation is then reduced into

$$\frac{h_{,\mathrm{n}}^2}{g_{\mathrm{nn}}} + \frac{h_{,1}^2}{g_{\mathrm{jj}}} = \frac{1}{g_{\mathrm{tt}}}\ . \qquad (77)$$

In the following, we will consider the simple case, when the metric coefficients have their Taylor expansions. Looking at the last equation, we can see, that the wavefront will have a Taylor expansion too. Hence, the general functions $\phi_k$ can be now specified, so that the expansion of wavefront becomes

$$h = \sum_{k=0}^{\infty} h_k(x^{\mathrm{n}})(g_{\mathrm{jj}}(\Delta x^{\mathrm{j}})^2)^k \ . \tag{78}$$

To demonstrate the further progress, we consider hereafter $g_{\mathrm{tt}}(x^{\mathrm{n}})$, $g_{\mathrm{jj}}(x^{\mathrm{n}})$, $g_{\mathrm{nn}}(x^{\mathrm{n}})$; this choice is general enough to provide all the results necessary in the scope of this work. Plugging the expansion (**78**) into eikonal equation (**77**) we obtain

$$\begin{aligned} 0=&\left(\frac{(h_0')^2}{g_{\mathrm{nn}}}-\frac{1}{g_{\mathrm{tt}}}\right)+\left(2\frac{h_0'(h_2 g_{\mathrm{jj}})'}{g_{\mathrm{nn}}}+4h_2^2 g_{\mathrm{jj}}\right)(\Delta x^{\mathrm{j}})^2+\\ &+\left(\frac{2h_0'(h_4 g_{\mathrm{jj}}^2)'+(h_2 g_{\mathrm{jj}})'^2}{g_{\mathrm{nn}}}+16h_2 g_{\mathrm{jj}}^2 h_4\right)(\Delta x^{\mathrm{j}})^4+\dots \end{aligned} \tag{79}$$

where prime indicates differentiation with respect to $x^{\mathrm{n}}$. Note, that all the quantities present are functions of $x^{\mathrm{n}}$ only. We observe, that the last equation can be sequentially solved order by order: indeed, in every higher order, just one new (unknown) coefficient appears. From the absolute term, we obtain

$$h_0 = \pm \int \sqrt{\frac{g_{\mathrm{nn}}}{g_{\mathrm{tt}}}}\, \mathrm{d}x^{\mathrm{n}} + c_0 \ .$$

Note that the last (indeterminate) integral is meant formally (i.e. there shall come no further integration constant from it), the same holds for all integrals, that shall appear in the expansion coefficients. Plugging now this expression into the nearest higher expansion into (**79**) we obtain

$$\pm\sqrt{\frac{g_{\mathrm{nn}}}{g_{\mathrm{tt}}}}\frac{(h_2 g_{\mathrm{jj}})'}{g_{\mathrm{nn}}} + 2h_2^2 g_{\mathrm{jj}} = 0 \ .$$

This ODE is of separate kind, which easily yields

$$h_2 = \pm \frac{1}{g_{\mathrm{jj}}\left(2\int \frac{\sqrt{g_{\mathrm{tt}} g_{\mathrm{nn}}}}{g_{\mathrm{jj}}}\, \mathrm{d}x^{\mathrm{n}} + c_2\right)} \ .$$

Similarly, these two coefficients can be plugged in the expression of nearest higher order. A linear ODE is obtained, which solved, gives

$$h_4=\pm\left[-2\int\frac{g_{tt}^2 g_{nn}\exp\left(2\int\frac{2g'_{jj}\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+g'_{jj}c_2+4\sqrt{g_{tt}g_{nn}}}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+c_2\right)}\mathrm{d}x^n\right)}{\sqrt{g_{tt}g_{nn}}g_{jj}^4\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+c_2\right)^4}\mathrm{d}x^n+c_4\right]\times$$

$$\times\exp\left(-2\int\frac{2g'_{jj}\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+g'_{jj}c_2+4\sqrt{g_{tt}g_{nn}}}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+c_2\right)}\mathrm{d}x^n\right)$$

and so on. Note that the behaviour with double sign is generic, so while the expansion is equal zero, we will in further without loss of generality use the positive one.

**Remembering the existive nature of optical axis definition, we can state, that in the case chosen, the optical axis always exists and the expansion**

$$h=\left(\int\sqrt{\frac{g_{nn}}{g_{tt}}}\mathrm{d}x^n+c_0\right)+\frac{1}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+c_2\right)}(g_{jj}\Delta x^j)^2+$$
$$+\left[-2\int\frac{g_{tt}^2 g_{nn}\exp(\chi)}{\sqrt{g_{tt}g_{nn}}g_{jj}^4\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+c_2\right)^4}\mathrm{d}x^n+c_4\right]\exp(-\chi)(g_{jj}\Delta x^j)^4+\ldots \quad (80)$$

**with**

$$\chi=2\int\frac{2g'_{jj}\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+g'_{jj}c_2+4\sqrt{g_{tt}g_{nn}}}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\mathrm{d}x^n+c_2\right)}\mathrm{d}x^n$$

**is valid, where $h$=const are the wavefronts.** The reason why our optical axis always exists lies in the choice of metric coefficients dependent only on $x^n$: in that simple case, there always exist a transformation of this coordinate, leaving the metric tensor in a form, from which it is clear, that the symmetry of the space chosen is sufficient for the axis to exist.

*To obtain the coefficients for the case of the flat spacetime with Cartesian coordinates, it is (for optical axis chosen as* $\boldsymbol{y=0}$*) enough to set* $x^{\mathrm{n}} \equiv x$ *and equal all metric coefficients one. Then, the wavefront is*

$$(x + c_0) + \frac{1}{2x + c_2} y^2 + \frac{-2x + c_4}{(2x + c_2)^4} y^4 \ldots = \text{const} . \qquad (81)$$

*In the polar coordinates,* $x^{\mathrm{n}} \equiv r$ *and the only other difference to previous case is that* $g_{\mathrm{jj}} = r^2$*. The wavefront expansion brought up is*

$$(r + c_0) + \frac{r}{-2 + rc_2} (\varphi - \varphi_0)^2 + \frac{\frac{2}{3} r + r^4 c_4}{(-2 + rc_2)^4} (\varphi - \varphi_0)^4 + \ldots = \text{const} . \qquad (82)$$

For the expansion of an axial wavefront within equatorial section of Schwarzschild geometry we then get

$$\begin{aligned} h = &[r + r_g \ln(r - r_g) + c_0] + \\ &+ \frac{r}{-2 + rc_2} (\varphi - \varphi_A)^2 + \frac{-\frac{1}{2} r_g + \frac{2}{3} r + r^4 c_4}{(-2 + rc_2)^4} (\varphi - \varphi_A)^4 + \ldots \end{aligned} \qquad (83)$$

Noting that (apart the signs) the solutions of equations determining the expansion coefficients were unique, we can observe that **the expansion form (of the axial wavefront) is characteristic for a spacetime** (more precisely of a given metric) and most importantly is thus independent of the particular wave chosen.

The only difference for different waves is lying within the specification of constants $c_k$, as in any aberration formulation theory should. We now show, how these constant can for particular waves be obtained. It is shown within ***The focus of a cluster and its aberrations***, how to obtain explicit parametric formulas ($r(\varrho)$, $\varphi(\varrho)$) for wavefronts from a ray equation (**13**), caustic (**14**) and eikonal along ray; the procedure is general.

Having such formulas – which as a parameter contain the ray coordinate $\varrho$ – we can plug them into general expansion (**80**) and re-expand into powers of $\varrho$. If the candidate was properly chosen, in every order of this new expansion, one constant $c_k$ can be precised.

*The spherical wavefronts* $\sqrt{(x - x_0)^2 + y^2} = \text{const}$*, centred in* $(x_0, 0)$*, can be for* $x \geq x_0$ *parametrised as* $(x_0 + \sqrt{\text{const}^2 - y^2},\ y)$*, whence the wavefronts itself are found determined by the constants*

$$c_0 = -x_0 \qquad c_2 = -2x_0 \qquad c_4 = 2x_0 \qquad \ldots \qquad (84)$$

*The same wavefronts in polar coordinates,* $\sqrt{r^2+x_0^2-2rx_0\cos\varphi}=\mathbf{const}$, *are determined in the appropriate expansion (***82***) by constants*

$$c_0 = -x_0 \qquad c_2 = \frac{2}{x_0} \qquad c_4 = -\frac{2}{x_0^3} \qquad \dots$$

The expansion coefficients before turning point for a testing field point source $[r_s{}', \varphi_A]$ in Schwarzschild background read

$$\begin{aligned} c_0' &= r_s' + r_g \ln(r_s' - r_g) \\ c_2' &= \frac{2}{r_s'} \\ c_4' &= \frac{1}{6}\frac{3r_g - 4r_s'}{r_s'^4} \\ &\vdots \end{aligned} \tag{85}$$

Note, that for true wavefronts **the constant numerating the wavefronts is never present in the expansion constants**. This is another usefulness of this approach, for in such a case the number of parameters is equal the number of ones that caustic has. In our two-dimensional case of interest, the constants are themselves parameter-less (as is also the caustic, as has been shown before).

Let us finally discuss the case, when the metric is sewed from more parts. We are interested in how the wavefront expansion coefficients will be modified. When the spacetime was covered by single metric, the constants $[x_0^{\mathrm{n}}, x_0^1] \equiv [x_s^{\mathrm{n}}, x_s^1]$ appearing within coordinate eikonal were directly the point source of radiation coordinates $[x_s^{\mathrm{n}}, x_s^1]$. Now, this will hold only within the (simply) connected region of metric validity, which contains the point source. In all other parts of the spacetime, it however still holds $[x_0^{\mathrm{n}}(x_s^{\mathrm{n}}, x_s^1, \rho), x_0^1(x_s^{\mathrm{n}}, x_s^1, \rho)]$. In this way, we have to use in every point of spacetime the appropriate metric wavefront expansion a get the knowledge of the constants relation to point source coordinates. The expansion treatment is not changed itself.

## The Wave-Aberration Coefficients

To discover the optical properties of the optical system, which is described by the axial wavefront expansion (**80**)

$$\boldsymbol{h}=\left(\int\sqrt{\frac{g_{nn}}{g_{tt}}}\,\mathrm{d}x^{n}+c_{0}\right)+\frac{1}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+c_{2}\right)}(g_{jj}\Delta x^{j})^{2}+$$
$$+\left[-2\int\frac{g_{tt}^{2}g_{nn}\exp\left(\chi\right)}{\sqrt{g_{tt}g_{nn}}g_{jj}^{4}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+c_{2}\right)^{4}}\,\mathrm{d}x^{n}+c_{4}\right]\exp\left(-\chi\right)(g_{jj}\Delta x^{j})^{4}+\ldots$$

with

$$\chi=2\int\frac{2g_{jj}'\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+g_{jj}'c_{2}+4\sqrt{g_{tt}g_{nn}}}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+c_{2}\right)}\,\mathrm{d}x^{n}$$

in the spacetime domain, where the metric whose coefficient are used is valid, we confront these waveefronts with another, refernce, system of (axial) wavefronts, which we describe generally by

$$\tilde{\boldsymbol{h}}=\left(\int\sqrt{\frac{g_{nn}}{g_{tt}}}\,\mathrm{d}x^{n}+\tilde{c}_{0}\right)+\frac{1}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+\tilde{c}_{2}\right)}(g_{jj}\Delta x^{j})^{2}+$$
$$+\left[-2\int\frac{g_{tt}^{2}g_{nn}\exp\left(\chi\right)}{\sqrt{g_{tt}g_{nn}}g_{jj}^{4}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+\tilde{c}_{2}\right)^{4}}\,\mathrm{d}x^{n}+\tilde{c}_{4}\right]\exp\left(-\tilde{\chi}\right)(g_{jj}\Delta x^{j})^{4}+\ldots$$

with

$$\tilde{\chi}=2\int\frac{2g_{jj}'\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+g_{jj}'\tilde{c}_{2}+4\sqrt{g_{tt}g_{nn}}}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}\,\mathrm{d}x^{n}+\tilde{c}_{2}\right)}\,\mathrm{d}x^{n}\,.$$

The crucial quantity is the *wave progress difference*, i.e. the difference $\delta = h - \tilde{h}$, which gives

$$\delta=(c_0-\tilde{c}_0)+\frac{\tilde{c}_2-c_2}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}dx^n+c_2\right)\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}dx^n+\tilde{c}_2\right)}(g_{jj}\Delta x^1)^2+ \\ +\left(\left[-2\int\frac{g_{tt}^2g_{nn}\exp(\chi)}{\sqrt{g_{tt}g_{nn}}g_{jj}^4\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}dx^n+c_2\right)^4}dx^n+c_4\right]\exp(-\chi)- \right. \\ \left. -\left[-2\int\frac{g_{tt}^2g_{nn}\exp(\tilde{\chi})}{\sqrt{g_{tt}g_{nn}}g_{jj}^4\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}dx^n+\tilde{c}_2\right)^4}dx^n+\tilde{c}_4\right]\exp(-\tilde{\chi})\right)(g_{jj}\Delta x^1)^4+\ldots \quad (\mathbf{86})$$

This quantity $\delta$ lies in the roots of the Gaussian optics - the aberration formulation. To observe it, we pick for the reference wavefronts $\tilde{h}$ the ones from a point source of general axial position $x^n = x^n_r$. This gives us two degrees of freedom: apart the source position, a phase constant $\tilde{c}$ of the reference wavefronts can be independently set so that $\delta = c - \tilde{c}$ held.

Then, we need not care about the absolute term in (**86**) and set the reference source position to annihilate the second-order term therein. The consequencies of such choice are much farer-going: now $\chi = \tilde{\chi}$, which let us write

$$h-\tilde{h}=(c_4-\tilde{c}_4)\exp\left(-2\int\frac{2g'_{jj}\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}dx^n+\omega g'_{jj}c_2+4\sqrt{g_{tt}g_{nn}}}{g_{jj}\left(2\int\frac{\sqrt{g_{tt}g_{nn}}}{g_{jj}}dx^n+\omega c_2\right)}dx^n\right)(g_{jj}\Delta x^1)^4+\cdots \quad (\mathbf{87})$$

The behaviour of the higher-order terms is similar: the condition, that the wavefronts are identical, only if their expansion coefficients coincide, is manifest.

*In the flat case, the wave progress difference reads*

$$\delta = \frac{c_4-\tilde{c}_4}{(2x+c_2)^4}y^4+\frac{4(c_4-\tilde{c}_4)(2c_2+c_4+\tilde{c}_4)+(c_6-\tilde{c}_6)(2x+c_2)}{(2x+c_2)^7}y^6+\ldots$$

The construction here provided is indeed in the roots of Gaussian optics, for choosing a point source wavefronts as reference ones gave rise to a *Gaussian focus* of the optical system - a point, from which the (curved) spherical waves emerge. Apart the expansion coefficients, a wavefront can be equivalently described by a set of constants $C_k = c_k - \tilde{c}_k$. If chosen $\tilde{c}_k$ for a point source wavefronts, emerging from Gaussian focus, the constants $C_k$ are the *aberration coefficients*. As can be seen, as we considered the axial wavefronts only, the aberration expansion contains terms corresponding to spherical aberrations of general orders only.

## The Caustic Expansion

Finally, let us return to the end of ***Part One***, where we tried to establish a connection between the caustic and the wavefronts. The problem was, that the form of wavefront cannot be considered known generally, hence the computations were interrupted. Now, however, we have the axial wavefront expansion generally available.

We adopt a bit different treatment, than was chosen before. Recall that there is a relation between wavefronts and ray families. We used it previously to determine the Laplacean of the wavefront with the knowledge of impulse representation eikonal only. Now, we make use this relationship from the other side: if $\boldsymbol{h}$**=const** are the wavefronts, then the corresponding rays $\boldsymbol{f}$**=const** must be given from

$$\mathrm{d}\tilde{f} = *\,\mathrm{d}h\ .$$

Recalling also that the caustic is invariant to diffeomorphisms of rays, we now need not seek the integrating factor and use any candidate for the ray family directly. Looking at last equation, from the same reason, any candidate for wavefronts can be used. Continuing, the caustic is given as a parameter derivative of ray family, hence, as long as the partial derivatives permutability, we have $\kappa=\partial\tilde{f}/\partial\varrho$ and thus,

$$\boldsymbol{\frac{\partial}{\partial\varrho} * \mathrm{d}h = 0}$$

is the equation of caustic, if the coordinate representation eikonal only is given. This is the sought for result.

# Conclusion

The main result of the work is the abstract construction of axial wavefronts expansion (**80**), which can obviously serve as a start to classification of optical systems within curved spacetimes. Based on the fact, that this expansion and the subsequent obtaining of wave aberrational expansion (**87**) are formed utilising the semi-geodesic coordinates, that are guaranteed on every Riemannean manifold, these results are very general. As a main result from the first part of the work, which successfully establishes the covariant formulation of geometrical optics within curved spacetimes may serve the coordinate eikonal Laplacean (**34**), found only with the knowledge of momentum representation formulas. Avoiding thus the need of Legendre transformation, other formulas and connections are provided as well. The application of the formulas obtained is shown clearly within the intermediate part of this work. As main applications, the original class of solutions (**62**) to Einstein equations that reproduces the Maxwell's fisheye analog in the scope of gravitational lensing and obtaining the focus (**66**) of the cluster and its aberrational structure (**68**),(**69**) as a GL example have to be considered.

As far as the author of this work is aware, all the results stated above are original.

Finally, let us note, that the field of gravitational lensing is well established domain of interest. The main publications and articles are cited throughout this work, let us however state here the total cumulative numbers for GL articles within past few years:

| **Years** | gravitational lensing | strong lensing | weak lensing | microlensing |
|---|---|---|---|---|
| 2000-2001 | 392 | 138 | 173 | 216 |
| 2001-2002 | 432 | 146 | 196 | 201 |
| 2002-2003 | 466 | 166 | 201 | 182 |
| 2003-2004 | 451 | 176 | 204 | 169 |
| 2004-2005 | 448 | 175 | 234 | 174 |

The results for the keywords stated in the table are taken from Citebase. Note, that the numbers are fuzzy-dependent. It can be seen that the strong lensing is a traditional discipline within gravitational lensing, whereas the weak one is on its rise; the period of maximal interest in microlensing seems to be slowly over. The overall number of publications for 'gravitational lensing' keyword in the interval of interest is 1303. The number of articles published by Czech authors in the same period is 5-10.

There is a hope, that the results within this work might help improve the situation with approximations, that so far have to superposed on the geometrical optics one.

# References


**[Alc]** ***Alcock, C. et al***:The MACHO Project: Microlensing Results from 5.7 Years of Large Magellanic Cloud Observations, ApJ **542**, 281-307, 2000

**[Arn]** ***Arnold V.I., Varchenko A.N., Gusein-Zade S.M.:*** Singularities of Differentiable Maps: Volume 1: The Classification of Critical Points, Caustics, Wave Fronts; Birkhauser 1985, ISBN 0817631879

**[Bar]** ***Bartelmann, M. Schneider, P.***: Physics Reports **340**, 291, 2001

**[Ber]** ***Berry, M.V. Upstill, C.***: Catastrophe optics: Morphologies of caustics and their diffraction patterns. Progress in Optics, XVIII:257–346, 1980

**[Bor]** ***Born M., Wolf E.:*** Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light; 7th edition, Cambridge University Press 1999, ISBN 0521642221

**[Bur]** ***Burke, W.L.***: Spacetime, Geometry, Cosmology, Univ Science Books 1980, ISBN 0935702016

**[Ein]** ***Einstein, A.***: Erklarung der Perihelbewegung des Merkur aus der allgemeinen Relativitatstheorie, Sitzungber. Preuss. kad. Wissensch. 1915, p. 831

**[Huc]** ***Huchra, J. et al***: 2237+0305: A new and unusual gravitational lens, Astron. J., **90**, 691

**[Koc]** ***Kochanek, C. et al***: Astroph. Journ., **445**, 549, 1995

**[Kor]** ***Korn, G.A. Korn, T.M.***: Mathematical Handbook for Scientists and Engineers : Definitions, Theorems, and Formulas for Reference and Review, Dover Publications 2000, ISBN 0486411478

**[Mol]** ***Mollerach S., Roulet E.:*** Gravitational Lensing and Microlensing, World Scientific Publishing Company 2002, ISBN 9810248520

**[Lan]** ***Landau, L.D., Lifshitz, E.M.*** : The Classical Theory of Fields, Fourth Edition : Volume 2 (Course of Theoretical Physics Series), Butterworth-Heinemann 1980, ISBN 0750627689

**[Lap]** ***Laplace, P.S.***: Exposition du systeme du monde, 1795

**[LePi]** ***Leonhardt, U. Piwnicki, P.***: Relativistic Effects of Light in Moving Media with Extremely Low Group Velocity, Phys. Rev. Lett. 84, 822-825 (2000).

**[Pet]** ***Petters A. O., Levine H., Wambsganss J.:*** Singularity Theory and Gravitational Lensing, Birkhauser 2001, ISBN 0817636684

**[Sei]** ***Seidel, L.*** :AN **43**, No 1027, S. 289, 1856

**[Sch]** ***Schneider P., Ehlers J., Falco E.E:*** Gravitational lenses, Springer 1999, ISBN 3540665064

[**Ste**] ***Stephani H., Kramer D. & al:*** Exact Solutions of Einstein's Field Equations; 2nd edition, Cambridge University Press 2002, ISBN 0521461367

[**Tho**] ***Thorn, K.S. Misner C.W. Wheeler, J.A.***: Gravitation, W. H. Freeman 1973, ISBN 0716703440

[**Wae**] ***van Waerbeke, L. et al***: Detection of correlated galaxy ellipticities on CFHT data: first evidence for gravitational lensing by large-scale structures, ASTRON.ASTROPHYS. 358 30 , 2000