

“Try to learn something about everything and everything about something ”

Thomas Henry Huxley (1825-1895)

Chapter 4

Interpolation and Approximation

The Weierstrass approximation theorem states that a continuous function $f(x)$ over a closed interval $[a, b]$ can be approximated by a polynomial $P_n(x)$, of degree n , such that

$$|f(x) - P_n(x)| \leq \epsilon, \quad x \in [a, b] \quad (4.1)$$

where $\epsilon > 0$ is a small quantity and n is sufficiently large. A polynomial representation is just one way of approximating a function. Approximation theory is concerned with finding various ways to represent a function over an interval and is not restricted to polynomial approximation.

The interpolation problem is the construction of a curve $y(x)$ which passes through a given set of data points (x_i, y_i) , for $i = 0, 1, \dots, n$ where the data points are such that $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$. The constructed curve $y(x)$ can then be used to estimate the values of y at positions x which are between the end points a and b (interpolation) or to estimate the value of y for x exterior to the end points (extrapolation).

Various industrial, business and research organizations routinely collect and analyze data. We shall investigate collected data in the form of two variables which we label x and y . We assume that the data can be labeled in some convenient way and represented in a tabular form. The table 4.1 illustrates one possible way of labeling and representing the data.

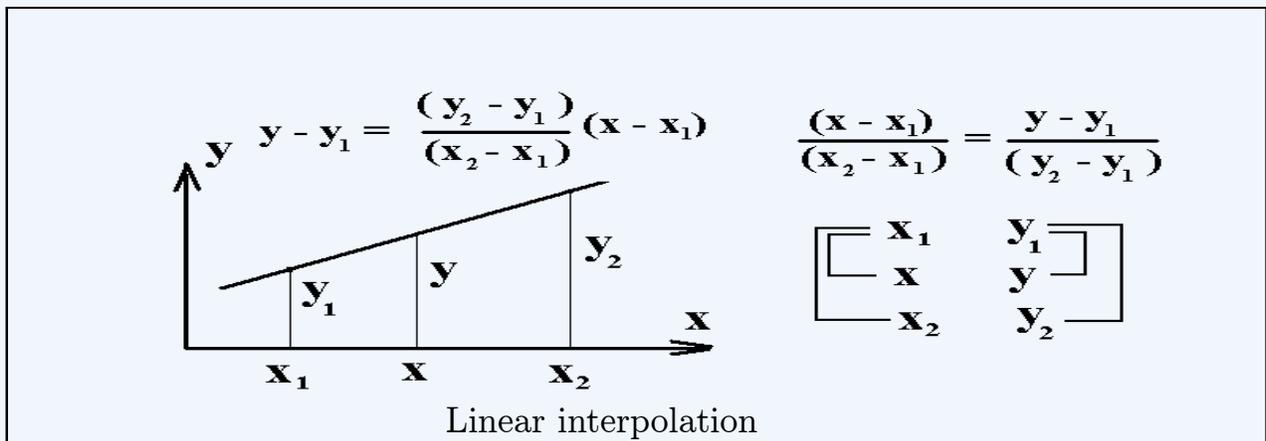
x	y
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3
\vdots	\vdots
x_{n-1}	y_{n-1}
x_n	y_n

Table 4.1 Tabular listing of (x, y) data values.

Whenever the data in table 4.1 is such that the independent variable x is evenly spaced, then the difference between any consecutive x -values has a constant value. We denote this constant value by h and write

$$\Delta x = x_{k+1} - x_k = h = \text{constant} \quad \text{for } k = 0, 1, 2, 3, \dots$$

Given a set of $(n + 1)$ data pairs (x_i, y_i) , $i = 0, \dots, n$, where the x_i values are equally spaced, we assume these (x, y) data pairs represent selected sample values $(x_i, y(x_i))$ from a continuous function $y(x)$, even though we do not know the function $y(x)$. We will study how to construct a polynomial $P_n(x)$, which satisfies $y_i = P_n(x_i)$ for $i = 0, 1, \dots, n$. This is called the interpolating polynomial which reproduces the function values at the given points x_i for $i = 0, 1, \dots, n$. The simplest polynomial interpolation is the straight line through two data points as illustrated in the figure.



The construction of a polynomial function $P_n(x)$ which satisfies $y_i = P_n(x_i)$ for $i = 0, 1, \dots, n$ has several purposes. First it can be used as an approximation function for reproducing the data values (x_i, y_i) for $i = 0, 1, \dots, n$. Secondly, polynomial interpolation is said to occur whenever one uses the approximating polynomial $P_n(x)$ to estimate the true y -value for a nontabulated x -value, where $x_0 \leq x \leq x_{n+1}$. Polynomial extrapolation is said to occur whenever one uses the approximating polynomial $P_n(x)$ to estimate the true y -values for x outside the interval $[x_0, x_{n+1}]$. We will use polynomial interpolation in later chapters to develop numerical techniques for differentiation and integration of a function. Polynomial interpolation will also arise in the development of numerical techniques for solving differential equations.

Difference Tables

We shall examine differences in the consecutive y -values associated with the table 4.1 representation of data. Define the first forward differences

$$\Delta y_1 = y_2 - y_1, \quad \Delta y_2 = y_3 - y_2, \quad \dots \quad \Delta y_i = y_{i+1} - y_i \quad (4.2)$$

and define second forward differences as differences of first forward differences. Second forward differences are written

$$\begin{aligned} \Delta^2 y_i &= \Delta(\Delta y_i) = \Delta(y_{i+1} - y_i) = \Delta y_{i+1} - \Delta y_i \\ &= (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) = y_{i+2} - 2y_{i+1} + y_i \end{aligned} \quad (4.3)$$

for $i = 1, 2, 3, \dots$. An $(n+1)$ -st ordered forward difference is defined as the difference of n -th ordered forward differences. Alternatively, one can define the stepping operator E defined by

$$E y_i = y_{i+1}, \quad E^2 y_i = y_{i+2}, \quad \dots, \quad E^n y_i = y_{i+n} \quad (4.4)$$

then the first and higher ordered forward differences can be written in an operator form. For example, since

$$\Delta y_i = y_{i+1} - y_i = E y_i - y_i = (E - 1) y_i \quad (4.5)$$

one can write $\Delta = E - 1$, and so the various forward differences can be expressed

first forward difference	$\Delta y_i = (E - 1) y_i = y_{i+1} - y_i$
second forward difference	$\Delta^2 y_i = (E - 1)^2 y_i = (E^2 - 2E + 1) y_i = y_{i+2} - 2y_{i+1} + y_i$
third forward difference	$\Delta^3 y_i = (E - 1)^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$
	\vdots
n -th forward difference	$\Delta^n y_i = (E - 1)^n y_i$

(4.6)

Using the binomial expansion one can verify that

$$\Delta^n y_i = y_{i+n} - \binom{n}{1} y_{i+n-1} + \binom{n}{2} y_{i+n-2} - \binom{n}{3} y_{i+n-3} + \cdots + (-1)^n y_i \quad (4.7)$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \quad (4.8)$$

are the binomial coefficients. One can now append columns of differences to the given data set with equal x -spacing to form a forward difference table. The subscript labeling of the points (x, y) in a difference table is arbitrary in that any point can be labeled (x_0, y_0) and the other points, as well as corresponding differences, are then labeled accordingly. The construction of a representative forward difference table associated with a constant x -value step size is illustrated in the table 4.2.

In the special case $\Delta x = h$ is constant, then the entries in the difference table can be scaled using the transformation

$$s = \frac{x - x_0}{h} \quad (4.9)$$

to obtain the scaled column of integer values listed in the table 4.2.

Table 4.2 Forward Difference Table								
s	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
-2	x_{-2}	y_{-2}						
			Δy_{-2}					
-1	x_{-1}	y_{-1}		$\Delta^2 y_{-2}$				
			Δy_{-1}		$\Delta^3 y_{-2}$			
0	x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		
			Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
1	x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
			Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$	
2	x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$
			Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
3	x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		
			Δy_3		$\Delta^3 y_2$			
4	x_4	y_4		$\Delta^2 y_3$				
			Δy_4					
5	x_5	y_5						

It is assumed that the data used to form the difference table is a discrete sampling from a function $y = y(x)$ which is continuous on some interval $[a, b]$. Therefore, one can apply the Weierstrass approximation theorem to construct an approximation of the function.

Note that if the data set is constructed from some polynomial, then the difference table will have the special property that the n th difference column will be all constants and so all columns of higher order differences will be zero. Whenever this occurs the data can be represented by a n th degree polynomial. Observe that

$$\Delta(x^n) = (x + h)^n - x^n = nhx^{n-1} + \text{lower order terms} \quad (4.10)$$

and for c_0 some nonzero constant one would have

$$\Delta(c_0x^n) = c_0nhx^{n-1} + \text{lower order terms.} \quad (4.11)$$

Consider the n th degree polynomial

$$P_n(x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n \quad (4.12)$$

where c_0, c_1, \dots, c_n are constants. Taking differences of this polynomial produces

$$\begin{aligned} \Delta P_n(x) &= c_0nhx^{n-1} + \text{lower order terms.} \\ \Delta^2 P_n(x) &= c_0n(n-1)h^2x^{n-2} + \text{lower order terms} \\ &\vdots \\ \Delta^n P_n(x) &= c_0n(n-1)(n-2)\cdots(3)(2)(1)h^n = c_0n!h^n = \text{constant} \\ \Delta^{(n+1)} P_n(x) &= 0 \end{aligned} \quad (4.13)$$

which demonstrates that for n -th degree polynomials, the n th differences are constant and the $(n+1)$ -st differences are zero.

Example 4-1. (Difference table.)

Form a forward difference table associated with the function $y = y(x) = x^3$ and the x -values 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4

Solution: We calculate the y -values and then the first, second, third and fourth forward differences to obtain the table 4.3.

Forward Difference Table					
x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.2	0.008				
		0.056			
0.4	0.064		0.096		
		0.152		0.048	
0.6	0.216		0.144		0
		0.296		0.048	
0.8	0.512		0.192		0
		0.488		0.048	
1.0	1.000		0.240		0
		0.728		0.048	
1.2	1.728		0.288		
		1.016			
1.4	2.744				

Table 4.3 Table of differences formed from the function $y = x^3$.

Whenever data pairs are collected from some sampling of an experiment there is usually errors associated with the data and in such cases the (x, y) data points will not give a forward difference table with a column of all constant values and so the data will not be a polynomial. However, by the Weierstrass approximation theorem, one can replace the true function $y = y(x)$ with some polynomial approximation $P_n(x)$. Let us investigate the construction of various polynomials from a selected set of data pairs.

Interpolating Polynomials

We wish to construct an approximating polynomial $P_n(x)$ which takes on the values y_0, y_1, \dots, y_n of $y(x)$ at the points x_0, x_1, \dots, x_n called nodes. If such a polynomial function can be constructed it is called an interpolation polynomial or collocation polynomial. The constructed polynomial function which passes through the given data points can be used to approximate $y(x)$ for any value of x over the interpolation interval (x_0, x_n) . If one uses the approximation polynomial to estimate values of $y(x)$ outside the interval (x_0, x_n) , then the process is called extrapolation. The process of extrapolation with polynomials is not recommended because polynomials $P_n(x)$ tend to oscillate between the values y_0, y_1, \dots, y_n when n is large and to diverge outside the interpolation interval. Whenever possible, interpolation is to be preferred over extrapolation when dealing with polynomials of high order.

One can construct an n th degree polynomial $y = P_n(x)$ which passes through $(n + 1)$ data points (x_i, y_i) for $i = 0, 1, 2, \dots, n$ and hence it can be called an interpolating polynomial. The polynomial constructed will be unique. To show uniqueness we employ the fundamental theorem of algebra which states that a polynomial of degree n has exactly n -roots. Now if we assume there are two different polynomials of degree n , say $y = P_n(x)$ and $y = \mathcal{P}_n(x)$ which have the same values at $(n + 1)$ data values (x_i, y_i) for $i = 0, 1, 2, \dots, n$, then the difference function $D(x) = P_n(x) - \mathcal{P}_n(x)$ is at most a polynomial of degree n which has $(n + 1)$ zeros. This can only occur if $D(x)$ is identically zero for all x values. Consequently, $P_n(x) = \mathcal{P}_n(x)$ and so the polynomials must be identically the same.

Note that polynomials can be represented in different ways. For example, the second degree polynomial $P_2(x) = x^2$ that passes through the points $(0, 0)$, $(1, 1)$ and $(2, 4)$ is unique, however, its representation is not unique and so the polynomial $P_2(x) = x^2$ can be represented in different ways. Four possible representations are

$$\begin{aligned} P_2(x) &= x(x - 2) + 2x & P_2(x) &= \frac{1}{2} + \frac{1}{2}(2x^2 - 1) \\ P_2(x) &= (x - 1)^2 - 2(x - 1) + 1 & P_2(x) &= x + x(x - 1) \end{aligned}$$

We now develop methods for construction of n th degree polynomials which collocate with the $(n + 1)$ data points $(x_0, y_0), \dots, (x_n, y_n)$.

Equally Spaced Data

Assume the x -values are equally spaced such that $x_i = x_0 + ih$ for $i = 0, 1, 2, \dots, n$, then the difference between any consecutive x -values is a constant and one can write $x_m - x_{m-1} = h$, for $m = 1, 2, \dots, n$. A polynomial representation of the form

$$P_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \dots + c_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

where the coefficients c_0, c_1, \dots, c_n are constants, and selected to make the polynomial produce the given data values, is required to satisfy the conditions

$$\begin{aligned} P_n(x_0) &= c_0 = y_0 \\ P_n(x_1) &= c_0 + c_1 h = y_1 \\ P_n(x_2) &= c_0 + c_1(2h) + c_2(2h^2) = y_2 \\ P_n(x_3) &= c_0 + c_1(3h) + c_2 6h^2 + c_3 6h^3 = y_3 \\ &\vdots \\ P_n(x_n) &= c_0 + c_1(nh) + c_2(n)(n-1)h^2 + \dots + n!c_n h^n = y_n. \end{aligned}$$

Solving for the coefficients we find

$$\begin{aligned}
 c_0 &= y_0 \\
 c_1 &= \frac{\Delta y_0}{h} = \frac{y_1 - y_0}{h} \\
 c_2 &= \frac{\Delta^2 y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} \\
 c_3 &= \frac{\Delta^3 y_0}{3!h^3} = \frac{y_3 - 3y_2 + 3y_1 - y_0}{3!h^3} \\
 &\vdots \\
 c_n &= \frac{\Delta^n y_0}{n!h^n}
 \end{aligned} \tag{4.14}$$

This produces the polynomial approximation

$$\begin{aligned}
 P_n(x) &= y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots \\
 &\quad + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1})
 \end{aligned} \tag{4.15}$$

which is called Newton's forward interpolation formula. Sometimes referred to as the Newton-Gregory forward interpolating polynomial. In terms of the scaled variable $s = \frac{x - x_0}{h}$, which has an integer value corresponding to each x_i data value, the equation (4.15) has the form

$$\begin{aligned}
 P_n(x) &= y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \\
 &\quad + \frac{s(s-1)(s-2) \cdots (s-n+1)}{n!}\Delta^n y_0
 \end{aligned} \tag{4.16}$$

which can also be represented in the form

$$P_n(x) = y_0 + \binom{s}{1}\Delta y_0 + \binom{s}{2}\Delta^2 y_0 + \binom{s}{3}\Delta^3 y_0 + \cdots + \binom{s}{n}\Delta^n y_0 \tag{4.17}$$

where the binomial coefficients $\binom{s}{i}$ multiplies the i th difference of y_0 from the difference table 4.2. The binomial coefficients $\binom{s}{i}$ represents the number of combination of s elements taken i at a time.

The lozenge diagram¹ illustrated in the figure 4-1 is often constructed as an aid to the representation of various polynomial interpolation formulas. One moves across the lozenge diagram from left to right in a sequence of straight

¹ Lozenge refers to the diamond shaped pattern created by diagonal lines.

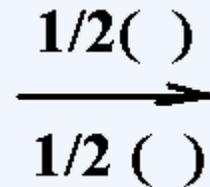
line paths. The lozenge diagram has certain rules for its use and within the lozenge diagram there are scale factors or coefficients used for construction of a collocation polynomial which is associated with a set of (x, y) data points. The polynomials are constructed as a series of terms and are produced as follows.

1. Move from left to right across the lozenge diagram starting with a value or modified value from the y -column.
2. One moves in a straight line path to the next column of the lozenge diagram. This straight line path can be either diagonally upward, horizontal or diagonally downward.
3. The straight line path points to a difference expression or binomial factor in the next column of the lozenge diagram. The quantity pointed to must be multiplied by a scale factor from the lozenge diagram to produce the next term in the series representing the interpolating polynomial. The scale factors are dependent upon the type of path selected. The following expressions are used as scale factors needed for the representation of the next term in the interpolation polynomial construction

(a) Use the coefficient below the path
if you move diagonally upward



(b) Use the average of the coefficients
above and below a horizontal path



(c) Use the coefficient above the path
if you move diagonally downward



For example, if we start at y_0 , then y_0 is a zeroth order polynomial approximation and represents the first term in the polynomial approximation which is given by $P_0(x) = y_0$. If we move horizontally from y_0 in a straight line to the right, we hit the term $\binom{s}{1}$ which must be multiplied by the scale factor $\frac{1}{2}(\Delta y_{-1} + \Delta y_0)$ to produce the first order approximation

$$P_1(x) = y_0 + \frac{1}{2}(\Delta y_{-1} + \Delta y_0) \binom{s}{1}, \quad s = \frac{x - x_0}{h} \quad (4.18)$$

Continue moving horizontally to the right we hit the next term $\Delta^2 y_{-1}$ in the lozenge diagram. This term must be multiplied by the scale factor $\frac{1}{2}[\binom{s+1}{2} + \binom{s}{2}]$

and so we produce the second order polynomial approximation

$$P_2(x) = y_0 + \frac{1}{2}(\Delta y_{-1} + \Delta y_0) \binom{s}{1} + \frac{1}{2} \Delta^2 y_{-1} \left[\binom{s+1}{2} + \binom{s}{2} \right], \quad s = \frac{x - x_0}{h} \quad (4.19)$$

Note that $P_1(x)$ and $P_2(x)$ depend upon the value of the ordinates y_{-1}, y_0 and y_1 centered about the point y_0 . By continuing our horizontal path to the right one can construct additional terms to add to the series and so produce higher order approximating polynomials. Observe that by moving further into the lozenge diagram one constructs higher degree interpolating polynomials which include the influence of additional ordinates surrounding the central point y_0 .

The lozenge diagram in figure 4-1 can be used to produce the following series in terms of the scaled variable $s = \frac{x - x_0}{h}$.

Newton's forward formula (path A-A)

$$P_n(x) = y_0 + \binom{s}{1} \Delta y_0 + \binom{s}{2} \Delta^2 y_0 + \binom{s}{3} \Delta^3 y_0 + \binom{s}{4} \Delta^4 y_0 + \cdots + \binom{s}{n} \Delta^n y_0$$

$$P_n(x) = y_0 + \sum_{i=1}^n \binom{s}{i} \Delta^i y_0 \quad (4.20)$$

Newton's backward formula (path B-B)

$$P_n(x) = y_0 + \binom{s}{1} \Delta y_{-1} + \binom{s+1}{2} \Delta^2 y_{-2} + \cdots + \binom{s+n-1}{n} \Delta^n y_{-n}$$

$$P_n(x) = y_0 + \sum_{i=1}^n \binom{s+i-1}{i} \Delta^i y_{-i} \quad (4.21)$$

Gauss backward formula (path C-C)

$$P_n(x) = y_0 + \binom{s}{1} \Delta y_{-1} + \binom{s+1}{2} \Delta^2 y_{-1} + \binom{s+1}{3} \Delta^3 y_{-2} + \binom{s+2}{4} \Delta^4 y_{-2} + \cdots$$

$$P_{2n}(x) = y_0 + \sum_{i=1}^n \left[\binom{s+i-1}{2i-1} \Delta^{2i-1} y_{-i} + \binom{s+i}{2i} \Delta^{2i} y_{-i} \right] \quad (4.22)$$

Gauss forward formula (path D-D)

$$P_n(x) = y_0 + \binom{s}{1} \Delta y_0 + \binom{s}{2} \Delta^2 y_{-1} + \binom{s+1}{3} \Delta^3 y_{-1} + \binom{s+1}{4} \Delta^4 y_{-2} + \cdots$$

$$P_{2n}(x) = y_0 + \sum_{i=1}^n \left[\binom{s+i-1}{2i-1} \Delta^{2i-1} y_{1-i} + \binom{s+i-1}{2i} \Delta^{2i} y_{-i} \right] \quad (4.23)$$