"One must learn by doing the thing; though you think you know it, you have no certainty until you try."

Sophocles (495-406)BCE

Chapter 9

Partial Differential Equations

A linear second order partial differential equation having a dependent variable u and two independent variables x, y, can be defined in terms of the linear partial differential operator $L(u)$ given by

$$L(u) = A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u$$

where A, B, C, D, E, F are coefficients which are real valued functions of the variables x and y. These coefficients are assumed to possess second derivatives which are continuous over a region R where the solution is desired. Linear second order partial differential equations of the form $L(u) = 0, \quad x, y \in R$ are called homogeneous equations and linear second order partial differential equations of the form $L(u) = G(x,y), \quad x, y \in R$ are called nonhomogeneous equations. Partial differential equation of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \quad x, y \in R$$

are called second order quasilinear partial differential equations. Note in the above definitions it is sometimes desirable, because of the physical problem being considered, to replace one of the variables x or y by the time variable t.

Some examples of linear second order partial differential equations are the Laplace equation

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad u = u(x,y) \qquad x, y \in R, \tag{9.1}$$

the one-dimensional heat equation

$$L(u) = \frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = G(x,t) \quad u = u(x,t), \quad t > 0, \quad 0 \le x \le L \tag{9.2}$$

where κ is a constant, and the one-dimensional wave equation

$$L(u) = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = G(x,t) \quad u = u(x,t), \quad t > 0, \quad 0 \le x \le L \tag{9.3}$$

where c is a constant.

Canonical Forms

Associated with both quasilinear and linear second order partial differential equations are canonical forms. These are special forms that the general form assumes under certain variable changes. The canonical forms are classified by the discriminant $\Delta = B^2 - AC$ formed from the A, B, C coefficients which multiply the highest ordered derivatives in the second order linear or quasilinear partial differential equation. In the partial differential equations that we shall consider, the discriminant is assumed to have a constant sign for all x, y in a region R of interest. The partial differential equation is called parabolic if $B^2 - AC = 0$ for all $x, y \in R$, it is called hyperbolic if $B^2 - AC > 0$ for all $x, y \in R$ and it is called elliptic if $B^2 - AC < 0$ for all $x, y \in R$. For example, the Laplace equation (9.1) has discriminant $\Delta = -1 < 0$ and is one of the canonical forms associated with elliptic partial differential equations. The heat equation (9.2) with discriminant $\Delta = 0$ is one of the canonical forms associated with parabolic equations and the wave equation (9.3) with discriminant $\Delta = c^2 > 0$ is one of the canonical forms associated with hyperbolic equations.

Boundary and Initial Conditions

Boundary conditions associated with a linear second order partial differential equation

$$L(u) = G(x, y) \quad \text{for} \quad x, y \in R$$

can be written in the operator form

$$B(u) = f(x, y) \quad \text{for} \quad x, y \in \partial R,$$

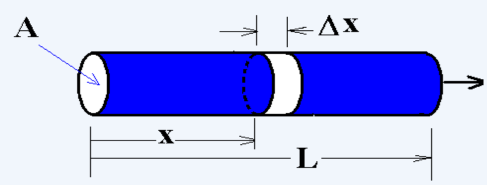
where ∂R denotes the boundary of the region R and $f(x, y)$ is a given function of x and y. If the boundary operator $B(u) = u$ the boundary condition represents the dependent variable being specified on the boundary. These type of boundary conditions are called Dirichlet conditions. If the boundary operator $B(u) = \frac{\partial u}{\partial n} = \operatorname{grad} u \cdot \hat{n}$ denotes a normal derivative, then the boundary condition is that the normal derivative at each point of the boundary is being specified. These type of boundary conditions are called Neumann conditions. Neumann conditions require the boundary to be such that one can calculate the normal derivative $\frac{\partial u}{\partial n}$ at each point of the boundary of the given region R. This requires that the unit exterior normal vector $\hat{n}$ be known at each point of the boundary. If the boundary operator is a linear combination of the Dirichlet and Neumann boundary conditions, then the boundary operator has the form $B(u) = \alpha \frac{\partial u}{\partial n} + \beta u$, where α and β are constants. These type of boundary conditions are said to be of the Robin type. The partial differential equation together with a Dirichlet boundary condition is sometimes referred to as a boundary value problem of the first kind. A partial differential equation with a Neumann boundary condition is sometimes referred to as a boundary value problem of the second kind. A boundary value problem of the third kind is a partial differential equation with a Robin type boundary condition. A partial differential equation with a boundary condition of the form

$$B(u) = \begin{cases} u, & \text{for } x, y \in \partial R_1 \\ \frac{\partial u}{\partial n}, & \text{for } x, y \in \partial R_2 \end{cases} \qquad \partial R_1 \cap \partial R_2 = \phi \quad \partial R_1 \cup \partial R_2 = \partial R$$

is called a mixed boundary value problem. If time t is one of the independent variables in a partial differential equation, then a given condition to be satisfied at the time $t = 0$ is referred to as an initial condition. A partial differential equation subject to both boundary and initial conditions is called a boundary-initial value problem.

The Heat Equation

The modeling of the one-dimensional heat flow in a thin rod of length L is accomplished as follows. Denote by $u = u(x, t)$ the temperature in the rod at position x and time t having units of [°C]. Assume the cross sectional area A of the rod is constant and consider an element of volume $d\tau = A\,\Delta x$ located between the positions x and $x + \Delta x$ in the rod as illustrated.



We assume the rod is of a homogeneous material and the surface of the rod is insulated so that heat flows only in the x-direction. Conservation of energy requires that the rate of change of heat energy associated with the volume element must equal the rate of heat energy flowing across the ends of the volume element plus any heat energy produced inside the element of volume. The physical properties of the rod are represented by the quantities:

c, the specific heat of the material with units [cal/g °C]

ϱ, the density of the material with units [g/cm^3]

k, the thermal conductivity of the material with units [cal/cm^2 sec °C/cm]

A, cross sectional area with units [cm^2]

The rate of change of heat stored in the volume element is given by

$$H_s = \frac{\partial}{\partial t}\int_x^{x+\Delta x} c\varrho A\, u(x,t)\, dx = \int_x^{x+\Delta x} c\varrho A\, \frac{\partial u(x,t)}{\partial t}\, dx.$$

Here $e = c\varrho u$ represents the thermal energy density of the volume element with units of [cal/cm^3]. The heat loss from the left and right ends of the volume element is found using the Fourier's law of heat flow which states that the heat flow normal to a surface is proportional to the gradient of the temperature. This heat loss can be represented

$$H_\ell = kA\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right],$$

where k is the thermal conductivity of the material. Let $H(x,t)$ denote the heat generated within the volume element with units of [cal/cm^3], then the heat generated by a source within the volume element can be represented

$$H_g = A\int_x^{x+\Delta x} H(x,t)\, dx.$$

The conservation of energy requires that $H_s = H_\ell + H_g$ or

$$\int_x^{x+\Delta x} c\varrho A \frac{\partial u(x,t)}{\partial t}\, dx = kA\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] + A\int_x^{x+\Delta x} H(x,t)\, dx. \tag{9.4}$$

The mean value theorem for integrals

$$\int_x^{x+\Delta x} f(x)\, dx = f(x+\theta\Delta x)\Delta x, \qquad 0 < \theta < 1$$

enables one to express the equation (9.4) in the form

$$kA\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] + AH(x+\theta_1\Delta x,t)\Delta x = c\varrho A\frac{\partial u(x+\theta_2\Delta x,t)}{\partial t}\,\Delta x. \tag{9.5}$$

Now divide by Δx and take the limit as $\Delta x \to 0$ to obtain the heat equation

$$k\frac{\partial^2 u}{\partial x^2} + H(x,t) = c\varrho\frac{\partial u}{\partial t}, \quad u = u(x,t), \quad 0 < x < L \tag{9.6}$$

or

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} + Q(x,t), \quad \text{where} \quad Q = \frac{H}{c\varrho}, \tag{9.7}$$

and $\kappa = \dfrac{k}{c\varrho}$ is known as the coefficient of thermal diffusivity with units of cm²/sec. The following table gives approximate values for the specific heats, density, thermal conductivity and thermal diffusivity of selected materials at 100 °C.

Coefficient	Fe	Al	Cu	Ni	Zn	Ag	Au
c=heat capacity [cal/g °C]	0.117	0.230	0.101	0.120	0.097	0.058	0.032
ρ= density [g/cm^3]	7.83	2.70	8.89	8.60	7.10	10.6	19.3
k=thermal conductivity [cal/sec cm^2 °C/cm]	0.107	0.490	0.908	0.138	0.262	0.089	0.703
κ=Thermal diffusivity [cm^2/sec]	1.168	0.789	1.011	0.134	0.381	0.145	1.138

A special case of equation (9.6) is when $H = 0$. One then obtains

$$k\frac{\partial^2 u}{\partial x^2} = c\varrho\frac{\partial u}{\partial t} \tag{9.8}$$

where $u = u(x,t)$ and k, c, ϱ are constants. This is known as the heat or diffusion equation. This type of partial differential equation arises in the study of diffusion type processes. It is classified as a parabolic equation.

An initial condition associated with the modeling of heat flow in a rod is written as $u(x,0) = f(x)$ where $f(x)$ is a prescribed initial temperature distribution over the rod. Dirichlet boundary conditions for the rod would be to specify the temperature at the ends of the rod and are written $u(0,t) = T_0$ and $u(L,t) = T_1$, where T_0 and T_1 are specified temperatures. (Recall the lateral surface of the rod is assumed to be insulated.) Boundary conditions of the Neumann type are written $-\frac{\partial u(0,t)}{\partial x} = g_0(t)$ and $\frac{\partial u(L,t)}{\partial x} = g_1(t)$ where $g_0(t)$ and $g_1(t)$ are specified functions of time t representing the heat flow across the boundary. If $g_0(t) = 0$, the boundary condition is said to be insulated so that no heat flows across the boundary. Robin type boundary conditions for the rod are expressed

$$-\alpha\frac{\partial u(0,t)}{\partial x} + \beta u(0,t) = g_0(t) \qquad \text{and} \qquad \alpha\frac{\partial u(L,t)}{\partial x} + \beta u(L,t) = g_1(t)$$

where again $g_0(t)$ and $g_1(t)$ are given functions of time t. These type of boundary conditions represent cooling or evaporation at the boundary.

Example 9-1. (Boundary-initial value problem)

As an example of a boundary-initial value problem consider the heat equation without heat sources which models the temperature distribution in a long thin rod of length L which is insulated along its length so that there is no heat loss. We must solve the partial differential equation (PDE)

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \quad u = u(x,t), \quad t > 0, \quad 0 < x < L, \quad K \text{ constant}$$

subject to both boundary and initial conditions.

A Dirichlet boundary-initial value problem for the heat equation has the form

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad 0 < x < L \\ \text{BC:} \quad & u(0,t) = T_0, \quad u(L,t) = T_L \\ \text{IC:} \quad & u(x,0) = f(x) \end{aligned}$$

Here the boundary conditions (BC) are the temperatures T_0 and T_L being specified at the ends of the rod for all values of the time t. The initial condition (IC) is that the initial temperature distribution $f(x)$ through the rod is being specified at time $t = 0$.

A Neumann boundary-initial value problem for the heat equation has the form

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad 0 < x < L \\ \text{BC:} \quad & -\frac{\partial u(0,t)}{\partial x} = \phi_0, \quad \frac{\partial u(L,t)}{\partial x} = \phi_L \\ \text{IC:} \quad & u(x,0) = f(x) \end{aligned}$$

Here the unit normal vectors to the ends of the rod are $\hat{n} = \hat{i}$ at $x = L$ and $\hat{n} = -\hat{i}$ at $x = 0$ and consequently the normal derivative at the end point boundaries are

$$\left.\frac{\partial u}{\partial n}\right|_{x=0} = \operatorname{grad} u \cdot \hat{n}\Big|_{x=0} = -\left.\frac{\partial u}{\partial x}\right|_{x=0} \quad \text{and} \quad \left.\frac{\partial u}{\partial n}\right|_{x=L} = \operatorname{grad} u \cdot \hat{n}\Big|_{x=L} = \left.\frac{\partial u}{\partial x}\right|_{x=L}$$

These terms represent the temperature gradient across the boundaries. Sometimes these are referred to as heat flow across the boundary since the heat flow is proportional to the gradient of the temperature. If u represents temperature [$^\circ C$], x represents distance [cm], then $\frac{\partial u}{\partial n}$ has units of [$^\circ C$/cm]. Note that the condition $\frac{\partial u}{\partial n} = 0$ denotes an insulated boundary. The Neumann boundary-initial value problem specifies the temperature gradients ϕ_0, ϕ_L across the boundaries as well as specifying the initial temperature distribution within the rod.

A Robin boundary-initial value problem for the heat equation has the form

$$\begin{array}{rl} \text{PDE:} & \dfrac{\partial u}{\partial t} - \kappa \dfrac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad 0 < x < L \\ \text{BC:} & -\dfrac{\partial u(0,t)}{\partial x} + hu(0,t) = \psi_0, \quad \dfrac{\partial u(L,t)}{\partial x} + hu(L,t) = \psi_L \\ \text{IC:} & u(x,0) = f(x) \end{array}$$

The boundary conditions represent the heat loss from the boundaries with h a heat loss coefficient which is a constant and dependent upon the rod material. In terms of diffusion processes the boundary conditions represent evaporation processes ψ_0, ψ_L specified at the ends $x = 0$ and $x = L$.

■

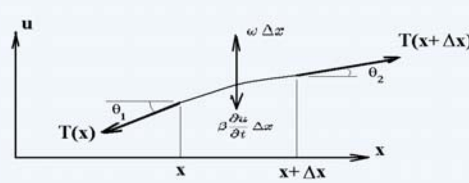
The wave equation

An example of a hyperbolic equation is the homogeneous partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{9.9}$$

where $u = u(x,t)$ and c is a constant. This is the wave equation of mathematical physics. It arises in the modeling of longitudinal and transverse wave motion. Some application areas where it arises are in the study of vibrating strings, electric and magnetic waves, and sound waves.

A mathematical model of a vibrating string is constructed by using Newton's laws and summing the forces acting on an element of string. Consider a section of string between x and $x + \Delta x$ as illustrated. Denote by $u = u(x,t)$ the string displacement [cm], at time t [sec], and introduce the additional symbols: ϱ [g/cm] to denote the lineal string density, $T(x)$ [dynes] the tension in the string at position x, ω [dynes/cm] an external force per unit length acting on the string. Further assume there exists a damping force proportional to the velocity $\frac{\partial u}{\partial t}$ of the string. This force is represented $\beta \frac{\partial u}{\partial t} \Delta x$ where β [dynes-sec/cm^2] denotes a linear velocity damping force per unit length. Assume there is equilibrium of forces in the horizontal direction. This requires



$$T(x + \Delta x)\cos\theta_2 = T(x)\cos\theta_1 = T_0 = \text{a constant.}$$

In the vertical direction we sum forces and apply Newton's second law to obtain

$$\begin{aligned}\varrho\Delta x\frac{\partial^2 u(x+\frac{\Delta x}{2},t)}{\partial t^2} =&T(x+\Delta x)\sin\theta_2 - T(x)\sin\theta_1 + \omega\Delta x - \beta\Delta x\frac{\partial u(x+\frac{\Delta x}{2},t)}{\partial t}\\ =&T_0(\tan\theta_2-\tan\theta_1)+\omega\Delta x-\beta\Delta x\frac{\partial u(x+\frac{\Delta x}{2},t)}{\partial t}\end{aligned} \tag{9.10}$$

Note that $\tan\theta_1 = \dfrac{\partial u(x,t)}{\partial x}$ and $\tan\theta_2 = \dfrac{\partial u(x+\Delta x,t)}{\partial x}$ so that when equation (9.10) is divided by Δx and one takes the limit as Δx approaches zero, there results the equation of motion of the vibrating string

$$\varrho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T_0\frac{\partial u}{\partial x}\right)+\omega-\beta\frac{\partial u}{\partial t}$$

In the special case $\beta = 0$, $\omega = 0$ and T_0 is constant, this reduces to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \quad 0 < x < L, \quad t > 0 \tag{9.11}$$

where $c^2 = T_0/\varrho$. is a constant. Here c has the units of velocity and denotes the wave speed. The wave equation (9.11) can be subjected to Dirichlet, Neumann or Robin type boundary conditions. Dirichlet conditions occur whenever one specifies the string displacements at the ends at $x = 0$ and $x = L$. Neumann conditions occur whenever the derivatives are specified at the ends of the string and Robin conditions occur when some linear combination of displacement and slope is specified at the string ends.

Elliptic equation

The elliptic partial differential equations that occur most frequently are the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 u = 0, \qquad u = u(x,y), \quad x,y \in R,$$

the Poisson equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = g(x,y) \quad \text{or} \quad \nabla^2 u = g(x,y) \qquad u = u(x,y), \quad x,y \in R,$$

(which is the nonhomogeneous form of Laplace's equation) and the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f(x,y)u = g(x,y) \quad \text{or} \quad \nabla^2 u+f(x,y)u = g(x,y) \qquad u = u(x,y), \quad x,y \in R.$$

These equations occur in a variety of applied disciplines.

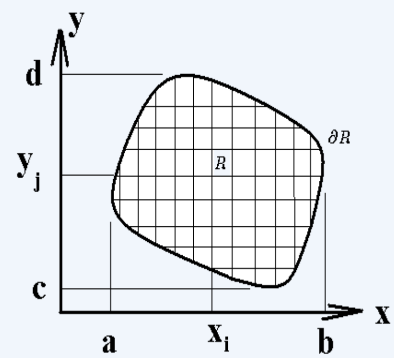
Examine the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad u = u(x,y,t) \quad x,y \in R$$

under steady state conditions where $\frac{\partial u}{\partial t} = 0$. If u does not change with time, then $u = u(x,y)$ and so the steady state heat equation is described by the Laplace equation subject to boundary conditions of the Dirichlet, Neumann or Robin type. Additional areas where the Laplace, Poisson and Helmholtz equation occur are potential theory, the study of torsion in cylindrical bars, and in the study of harmonic functions.

Numerical solution of the Laplace equation

Consider the boundary value problem to solve either the Laplace equation



$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x,y \in R$$

or the Poisson equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = h(x,y) \quad x,y \in R$$

where R is a simply-connected region† of the x,y plane. We begin by seeking solutions u which are subject to Dirichlet boundary conditions of the form $u\mid_{x,y\in\partial R} = g(x,y)$, where ∂R denotes the boundary of the region R and $g(x,y)$ is a given function of x and y to be evaluated at points on the boundary. The general technique for obtaining the numerical solution of the above equations is as follows. The region R is divided up into some form of a grid, mesh or lattice structure involving a discrete set of points or nodes (x_i, y_j) for $i = 0,1,2,\ldots,n$ and $j = 0,1,2,\ldots,m$. The partial derivatives are then approximated by various differences in terms of the functional values $u_{i,j} = u(x_i, y_j)$ at the grid or node points. These difference approximations for the derivatives are then substituted into the partial differential equation. In this way the partial differential equation is reduced to a discretized form. This discretized form of the partial differential equation represents a difference equation which is to be applied over each of the nodal points of the grid structure. This produces a system of linear equations with unknowns $u_{i,j} = u(x_i, y_j)$ that must be solved for. Any of the previous numerical methods for solving linear systems can be applied to solve the resulting

† A simply-connected region is such that any simple closed curve within the region can be continuously shrunk to a point without leaving the region.

system of equations. Having obtained the solution values $u_{i,j}$, these discrete values can be used to approximate the true solution $u = u(x, y)$ of the given partial differential equation. If values of $u = u(x, y)$ are desired at nonlattice points, then one must use interpolation to obtain the approximated values.

Sometimes it is assumed that the addition of more lattice or mesh points, to produce a finer grid structure, will give an improved approximation to the true solution. However, this is not necessarily a true statement. As the grid structure gets finer and finer there are produce many more nodal points to solve for and the resulting system of equations can become quite large. This requires more computation to calculate the solution and consequently there is the increased risk of round off error build up. Somewhere between a coarse and fine grid structure lies the optimal grid structure which minimizes both approximation error and round off error in computing the solution.

In general, given a partial differential equation, where the solution is desired over a region R which is rectangular in shape, one can divide the region into a rectangular grid by defining a Δx and Δy spacing given by

$$h = \Delta x = \frac{b-a}{n}, \qquad k = \Delta y = \frac{d-c}{m}.$$

One can then write $x_i = x_0 + ih$ and $y_j = y_0 + jk$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ where $x_0 = a$, $y_0 = c$. We use the notation $u_{i,j} = u(x_i, y_j)$ to denote the value of u at the point (x_i, y_j) and then develop approximations to the various partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ and higher derivatives, occurring in the partial differential equation. We evaluate the given partial differential equation at the point (x_i, y_j) and then substitute the partial derivative approximations to obtain a difference equation. The various partial derivative approximations involve the step sizes h and k and combinations of the grid points in the neighborhood of (x_i, y_j). These derivative approximations can be developed by manipulation of the various Taylor series expansion of a function of two variables about the point (x_i, y_j). It is left as an exercise to derive these various derivative approximations. A variety of these derivative approximations can be found in the exercises at the end of this chapter.

For the Laplace equation we begin with a 5-point formula to approximate the derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial u^2}$ and leave more complicated approximations for the exercises.