STELLAR DYNAMICS

- CONCERN MOTION OF STARS IN GRAV. SY. GALAXIE, GLOB. CLUSTERS, STELLAR CL.

B1 CLASSICAL GRAVITY

$\Delta\phi = 4\pi G \cdot \rho$ POISSON EQUATION, FIELD EQUATION FOR CLASSICAL GRAVITY

↳ LINEAR POTENTIAL - CAN JUST ADD UP SOL.

- NOW LETS THINK WHERE THIS EQ. MIGHT FAIL - NO TIME EVOLUTION, JUST 3D LAPLACIAN, NO DYNAMICAL PROPERTY IN IT SELF (DOES NOT HAVE GRAV. FIELD), BUT GRAVITATIONAL WAVES!!!

LET'S DERIVE THE MOST GENERAL GRAVITATIONAL THEORY IN CLASICAL NON-RELATIVISTIC ~~p~~ GRAV. THEORY:

- <u>LAGRANGE FORMALISM</u> AND WE USE ~~...~~ HAMILTON PRINCIPL:

$\delta S = 0$ ... $S = \int d^3x\, L$ LEAST ACTION

LAGRANGE DENSITY - DYNAMICS OF SYSTEM, WE WILL TRY TO WRITE DOWN SOMETHING THAT IS INVARIANT ~~a~~ UNDER ROTATION, AND HAS AT MOST SQUARES

ϕ - POT. IS INVARIANT (SCALAR FUNCT.)

$$L = \underbrace{\frac{1}{2}(\nabla\phi)^2}_{} + \lambda\phi + \frac{m}{2}\phi^2 + 4\pi G\rho\phi$$

↳ GRADIENT OF A FIELD (INVARIANT UNDER ROTATION) EQUIVALENT TO $\frac{1}{2}\phi\Delta\phi$ BY INTEGRATION BY PARTS

ALL TERMS ARE (1) PARITY INVARIANT → ISOTROPIC POTENTIAL

(2) AT MOST SQUARES IN ϕ → LINEAR FIELD EQUAT.

NOW WE PERFORM VARIATION UNDER THIS <u>L</u> EXPRESION:

$$\delta S = \int d^3x \left(\frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial\nabla\phi}\delta\nabla\phi\right) = \int d^3x \underbrace{\left(\frac{\partial L}{\partial\phi} - \nabla\frac{\partial L}{\partial\nabla\phi}\right)}_{0}\delta\phi = 0$$

EULER-LAGRANGE EQUATION $\dfrac{\partial L}{\partial\phi} = \nabla\dfrac{\partial L}{\partial\nabla\phi}$ APPLY TO

$$L = \frac{1}{2}(\nabla\phi)^2 + \lambda\phi + \frac{m}{2}\phi^2 + 4\pi G\rho\phi$$

$$(\Delta - m^2)\phi = 4\pi G \cdot \rho + \lambda$$

SCALE ≥ 1 NOT IN GALAXY!

COSMOL. KONS - PLAYS ROLE ON LARGE SCAL

REQUIRE: (3) POWER LAW, SCALE FREE SOLUTIONS: $\rightarrow m=0$

(4) ATTRACTIVE GRAVITY, ONLY if $\rho \neq 0 \rightarrow \lambda = 0$

$\rightarrow$ POISSON EQUATION $\Delta\phi = 4\pi G \cdot \rho$

λ - COSM. KONST. - VERY SMALL, NOT ZERO

LINEARITY: BUILT-UP ANY CHARGE DISTRIBUTION FROM POINT CHARGES, FINDING A FIELD ~~EQUATION~~ POTENTIAL DISTRIBUTION SEPARATES INTO TWO PARTS, FIRST WE NEED TO KNOW WHAT POTENTIAL POINT CHARGES GENERATE AND THAN YOU NEED TO KNOW HOW CAN YOU ASSEMBL THE ENTIRE POTENTIAL FROM THE INDIVIDUAL POINTS.

GREEN FUNCTION!

SIMPLE PROBLEM

$$\Delta G(\vec{r};\vec{r}\,') = 4\pi\,\delta_D(\vec{r}-\vec{r}\,')$$

GREEN FUNCTION - POTENTIAL FOR A POINT CHARGE

$$G(\vec{r};\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}$$

LET'S MULTIPLY IT BY $\rho(\vec{r}\,')$

$$\Delta\int_V d^3r'\, G(\vec{r};\vec{r}\,')\cdot\rho(\vec{r}\,') = \Delta\int_V d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = 4\pi\int d^3r'\,\rho(\vec{r}\,')\,\delta_D(\vec{r}-\vec{r}\,')$$

$$\Delta G = 4\pi\,\delta_D(\vec{r}-\vec{r}\,')$$

$$\phi(\vec{r}) = 4\pi\cdot\rho(\vec{r})$$

B2 CLASSICAL GRAVITATIONAL FIELDS

- ISOTROPIC - Δ LAPLACIAN
- IN TERMS OF r

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho \quad ; \quad \rho \sim \text{SPHERICALLY SYMETRIC}$$

GRAV. ACCEL - SLOPE OF ϕ

$$\rightarrow g_r = \frac{G}{r^2}\cdot\underbrace{\int 4\pi\cdot r^2\,dr\,\rho}_{\text{ASSUME ENCLOSED MASS} = M} = \frac{GM}{r^2} = -\nabla\phi$$

(ASSUME SPHERICAL SYMETRY)

$\nabla\phi$ $\rightarrow$ GRADIENT OF GRAVITATIONAL POTENTIAL.

GRAV. ACCELERATION $g_r \rightarrow$ IS A VECTOR FIELD

GEOMETRIC PICTURE $\xrightarrow{\text{IS}}$ FLUX OF FIELD LINES → CONSTANT FLUX & MASS

$$\int d^3r \; \mathrm{div}\, \vec{g} = \int d\vec{A} \cdot \vec{g} = \vec{r}^2 \cdot \int d\Omega \underbrace{\vec{e}_r \cdot \vec{g}}_{= g_r} = r^2 \frac{GM}{r^2} = GM$$

g (THATS THE LINES)

IF THERE IS A MASS, THERE ARE FIELD LINES COMING OUT.
FLUX OF FIELD LINES IS PROPORTIONAL OF THE MASS.
WHAT ARE THE SUITABLE INTEGRATION BOUNDS?
ACTUALLY $\delta\phi = G \cdot \int_{r_1}^{r_2} \frac{dr}{r^2} \cdot \int 4\pi r'^2 dr' \, \rho(r')$

$\delta\phi$ IS THE DIFFERENCE IN POTENTIAL BETWEEN r_1 & r_2 → RELATIVE ACCELERATION δg_r

GREEN FUNCTION ~ POTENTIAL FOR A POINT CHARGE.
$\Delta\phi = 4\pi G \, \delta_D(r)$ (PUT CHARGE TO $r' = 0$)

$\phi = \frac{1}{r}$: $\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{\partial}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0$ if $r \neq 0$

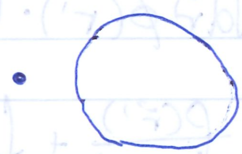
if $r = 0$: $\Delta\phi = \mathrm{div}\nabla\phi = \mathrm{div}\,\vec{g} \rightarrow$

$\rightarrow \int d^3r \; \mathrm{div}\nabla\phi = \int d\vec{A}\,\nabla\phi = \int d\Omega \cdot r^2 \cdot g_r = \underline{\underline{4\pi}}$

$\nabla\phi = \frac{1}{r^2}$

- WHEN OUTSIDE OF POSITION OF A CHARGE POTENTIAL IS ZERO
- IF I'M IN THE POSITION OF A CHARGE THEN $\neq 0$

CONSTANT FLUX IF CHARGE IS INSIDE



NO FLUX IF CHARGE OUTSIDE

$\Delta\phi$ - SCALAR FUNCTION

B3 SOLVING LINEAR POTENTIAL PROBLEMS

$$\phi(r) = G\int d^3r \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$, AS A SOLUTION TO $\Delta\phi = 4\pi G\cdot\rho$

IDEA: A LINEAR FIELD THEORY (NEWTONIAN GRAVITY OR MAXWELL ELECTRODYNAMICS) ALLOWS TO FIND THE SOLUTION TO $\phi(\vec{r})$ FROM A SUPERPOSITION OF THE COULOMB-FIELDS FROM THE INDIVIDUAL CLOSE ELEMENTS MAKING UP $\rho(\vec{r})$.

2nd GREEN THEOREM

$$\vec{M} = \varphi\nabla\psi \; ; \; \mathrm{div}\,\vec{M} = \nabla\varphi\cdot\nabla\psi + \varphi\Delta\psi$$

$$\int_V d^3r \; \mathrm{div}\,\vec{M} = \int_{\partial V} d\vec{A}\cdot\vec{M}$$

$$\rightarrow \int_V d^3r\left[\nabla\varphi\nabla\psi + \varphi\Delta\psi\right] = \int_{\partial V} d\vec{A}\cdot\varphi\nabla\psi = \int_{\partial V} dA\cdot\varphi\frac{\partial\psi}{\partial n} \quad \frac{\partial\psi}{\partial n} = \vec{n}\cdot\nabla\psi$$

INTERCHANGE φ AND ψ: AND SUBTRACT

$$\int_V d^3r\left[\varphi\Delta\psi - \psi\Delta\varphi\right] = \int_{\partial V} dA\left[\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right]$$

$$\psi = \frac{1}{|r-r'|} \longrightarrow \Delta\psi = 4\pi\delta_D(r-r')$$

$$\varphi = \phi(r') \longrightarrow \Delta\phi = \Delta\varphi = 4\pi G\cdot\rho(r)$$

$$\int_V d^3r'\varphi\Delta\psi = \int d^3r\,\phi(r')\cdot 4\pi\,\delta(\vec{r}-\vec{r}') = 4\pi\,\phi(r') =$$

$$= \int d^3r'\cdot 4\pi G\frac{\rho(\vec{r}')}{|r-r'|} + \int_{\partial V} dA\;\phi\cdot\frac{\partial}{\partial n}\frac{1}{|r-r'|} - \frac{1}{|r-r'|}\frac{\partial}{\partial n}\phi$$

$$\rightarrow \phi(\vec{r}) = G\int_V d^3r'\frac{\rho(\vec{r}')}{|r-r'|} + \int_{\partial V} dA\left[\underbrace{\phi\frac{\partial}{\partial n}\frac{1}{|r-r'|}}_{\text{DIRICHLET}} - \underbrace{\frac{1}{|r-r'|}\frac{\partial}{\partial n}\phi}_{\text{NEUMANN}}\right]$$

B4 SPHERICAL MULTIPOLE EXPANSION

$$\phi(\vec r) = G \cdot \int d^3 r' \, \frac{\rho(\vec r')}{|\vec r - \vec r'|}$$

WE ARE INTERESTED IN POTENTIAL AT LARGE DISTANCES $|\vec r| \gg |\vec r'|$ (FAR AWAY FROM THE SOURCE)

$$\frac{1}{|\vec r - \vec r'|} = \frac{1}{\sqrt{r^2 - 2 r r' \cos\gamma + r'^2}} = \frac{1}{r} \frac{1}{\sqrt{1 - 2\frac{r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2}} \cong$$

TAYLOR EXPAN...

$$\cong \frac{1}{r} \cdot \left(1 + \cos\gamma \cdot \frac{r'}{r} + \left(\frac{3}{2}\cos^2\gamma - \frac{1}{2}\right)\left(\frac{r'}{r}\right)^2 + \dots\right)$$

$$= \frac{1}{r}\left(P_0(\cos\gamma) + P_1(\cos\gamma)\frac{r'}{r} + P_2(\cos\gamma)\cdot\left(\frac{r'}{r}\right)^2 + \dots\right)$$

$$= \frac{1}{r}\sum_l P_l(\cos\gamma)\cdot\left(\frac{r'}{r}\right)^{l}$$ l — L MALÉ L

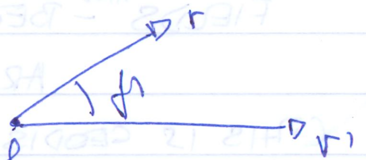
NOW WE INSERT ADDITION THEOREM OF THE SPHERICAL HARMON.

LEGANDROVY POLYN.

$$P_l \cos\gamma = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_{lm}(\hat r)\, Y^*_{lm}(\hat r')$$

WE CAN SUBSTITUTE THIS

DIRECTION OF $\vec r$



$$\phi(\vec r) \cong G \cdot \int_V d^3 r' \cdot \rho(\vec r') \frac{1}{r} \sum_l \left(\frac{r'}{r}\right)^l \cdot \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat r)\, Y^*_{lm}(\hat r') =$$

$$= G \sum_l \sum_m \underbrace{\sqrt{\frac{4\pi}{2l+1}} \cdot \frac{Y_{lm}(\hat r)}{r^{l+1}}}_{r \,\&\, r' \text{ ARE SEPARATED}} \underbrace{\sqrt{\frac{4\pi}{2l+1}} \int_V d^3 r' \cdot \rho(\vec r')\, r'^l \cdot Y^*_{lm}(\hat r')}_{Q_{lm}}$$

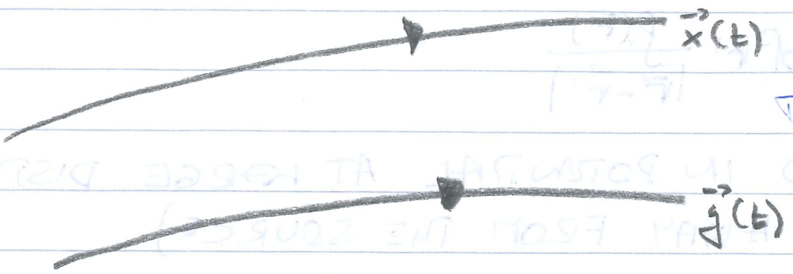
$$\phi(\vec r) = G \cdot \sum_l \sum_m \sqrt{\frac{4\pi}{2l+1}} \cdot Y_{lm}(\hat r) \cdot Q_{lm} \cdot \frac{1}{r^{l+1}}$$

Q_{lm} IS THE MULTIPOLE MOMENT OF ORDER $(l; m)$

IT CONTRIBUTES TO THE POTENTIAL WITH A TERM $\propto \frac{1}{r^{l+1}}$

FAR FROM THE SOURCE, ONLY LOW l-MOMENTS ARE RELEVANT

B5 NON-RELATIVISTIC MOTION IN A CLASSICAL FIELD



TWO OBJECTS IN GRAV. FIELD

NEWTON'S EQUATION OF MOTION OF THESE TWO OBJECTS $\ddot{x}_i = -\partial_i \phi \; ; \; \ddot{y}_i = -\partial_i \phi$

RELATIVE MOTION $\delta_i = x_i - y_i \qquad \ddot{\delta}_i = \ddot{x}_i - \ddot{y}_i$ (LINEAR)

THE RELATIVE DISTANCE BETWEEN TWO OBJECTS MATTER:

TIDAL FORCE: $\partial_i \phi |_y = \partial_i \phi |_x + \partial_i \partial_j \phi |_x (y - \cancel{g} x)_j + \dots$

$$\to \ddot{\delta}_i = \ddot{x}_i - \ddot{y}_i = -\partial_i \phi |_x + \partial_i \phi |_y = \partial_i \partial_j \phi |_x (y - x)_j =$$

$$= -\partial_i \partial_j \phi \, \delta_j$$

RELATIVE ACCELERATION DUE TO TIDAL FORCE $= (-\partial_i \partial_j \phi)$

(IS PROPORTINAL)

IMPORTANT - ELIPTICAL GALAXIES ARE SQUIZED BY GRAV. FIELDS AND DESTROYED - BECAUSE INDIVIDUAL POINTS OF THESE GALAXIES ARE ACCELERATED AT DIFFERENT RATES.

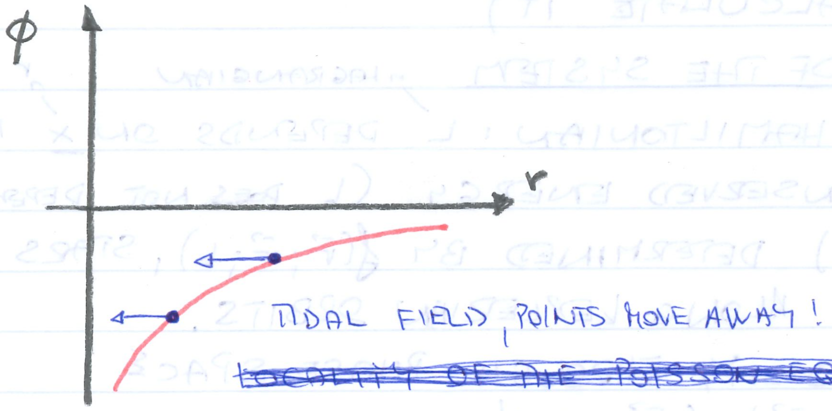
(THIS IS GEODISIC DEVIATION IN A CLASSICAL CONTEXT)

TRAJECTORIES ARE CURVED TOWARDS EACH OTHER IF MATTER IS IN BETWEEN.

$$\Delta \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = 4\pi G \rho$$

$$\frac{\partial \phi}{\partial r} = \frac{G}{r^2} \int_0^R 4\pi r^2 dr \, \rho \quad \to \quad \frac{\partial^2 \phi}{\partial r^2} = -\frac{2GM}{r^3} < 0$$

NEGATIVE CURVATURE, POINTS MOVE AWAY!



B6 COLLISIONLESS SYSTEMS - BOLTZMANN EQUATION

- HOW WOULD MASS OF STARS MOVE IN A POTENTIAL THAT IS POSSIBLY GENERATED BY THEM SELVES. (GALAXIE, GLOB. CLUSTERS)
- WHY NOT GALAXY COLAPS ON ITS OWN WEIGHT, STARS PERFORM MOTION AND WE CAN SETUP A STABLE CONFIGURATION, WHERE THE STARS ARE IN MOTION AND WE GET COUNTER REACTION WITH RESPECT TO GRAVITY.
- WE CAN MEASURE MASS OF SELF GRAVITATING SYSTEMS, THANKS TO OBSERVATION
- IN THIS LECTURE WE WILL LOOK FOR STABLE CONFIG. IN POSITION AND M.— SPACE
- COLLISION LESS SYSTEMS: STARS IN A GALAXY OR IN A CLUSTER NEVER COLLIDE (NEAR SEPARATION >> SIZE OF A STARS), THEY MOVE IN A SMOOTH GRAVITATIONAL POTENTIAL (GENERATED BY THE STARS THEMSELF + BY CDM)
- COLLISIONLESS BOLTZMAN EQ. - DESRIBES HOW STARS IN VELOCITY SPACE AND POSITION SPACE EVOLVE IN TIME (IF STARS DO NEWTONIAN MOTION)

POTENTIAL $\phi(\vec{r};t) \longleftarrow$ SET UP INSTANTANEUSLY, $\Delta\phi = 4\pi G\rho$

PHASE SPACE DENSITY $f(\vec{r};\vec{v};t)$ → PROBALITY OF FINDING AN OBJECT AT POSITION $\vec{r}$ WITH VELOCITY $\vec{v}$ AT TIME t

$$f(\vec{r};\vec{v};t) \geq 0 \;, \int d^3r \int d^3v \, f(\vec{r};\vec{v};t) = 1$$

(HOW TO CALCULATE IT)

TIME EVOLUTION OF THE SYSTEM

- SYSTEM IS HAMILTONIAN: L DEPENDS ON $\underline{x}$ AND $\dot{x}$ → (LAGRANGIAN; r, v)
  → CONSERVED ENERGY (L DOES NOT DEPEND ON TIME ITSELF)
- $f(\vec{r};\vec{v};t+\Delta t)$ DETERMINED BY $f(\vec{r};\vec{v};t)$, STARS MOVE ALONG KEPLERIAN ORBITS.
- FLOW OF PROBABILITY THROUGH PHASE SPACE
  $\partial_t \vec{w} = \partial_t(\vec{r};\vec{v}) = (\vec{v}; -\nabla\phi)$ ~ FLUX WITH PHASE SPACE COORDINATES $(\vec{r};\vec{v})$
- CONTINUITY - NO STARS GET LOST IN TIME EVOLUTION

$$\underbrace{\partial_t f}_{\text{TIME EVOLUTION OF } f \text{ (DENSITY)}} + \sum_\alpha \frac{\partial}{\partial w_\alpha}\underbrace{(f\dot{w}_\alpha)}_{\text{FLUX OF STARS IN PHASE SPACE}} = 0$$

CONTINUITY EQUATION "PRESERVED"

THIS THE "DIVERGENCE" SORT OF "OF FLUX"

$$\sum_\alpha \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_i \left(\underbrace{\frac{\partial v_i}{\partial x_i}}_{=0} + \frac{\partial \dot{v}_i}{\partial v_i}\right) = \sum_i -\frac{\partial}{\partial v_i}\left(\frac{\partial\phi}{\partial x_i}\right) = 0$$

NEWTONIAN EQ. OF MOTION

~~INDEPENDENT~~ PHASE SPACE COORDINATES = VELOCITY COOR. AND POSITION COORD. ARE INDEPENDENT ON EACH OTHER

ϕ DOES NOT DEPEND ON VELOCITY (NEWTONIAN POTENTIAL)

COLLISION LES BOLTZMAN EQUATION

$$\partial_t f + \sum_\alpha \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0 \quad \text{BECAUSE} \quad \frac{\partial}{\partial v_i}\frac{\partial\phi}{\partial x_i} = \frac{\partial}{\partial x_i}\cdot\frac{\partial\phi}{\partial v_i} = 0$$

AGAIN COLL. BOLTZMAN EQUATION ~~WITH~~ - WE CAN EXPANDED

$$\frac{\partial}{\partial t} f + \sum_\alpha \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0$$

SPLIT IT UP INTO VELOCITY AND SPACE PART.

$$\frac{\partial}{\partial t} f + \sum_i \underbrace{\dot{x}_i}_{v_i}\frac{\partial f}{\partial x_i} + \underbrace{\dot{v}_i}_{\frac{\partial v_i}{\partial t} \to -\frac{\partial\phi}{\partial x_i}}\frac{\partial f}{\partial v_i} = \frac{\partial f}{\partial t} + \underbrace{\sum_i v_i\frac{\partial f}{\partial x_i}}_{\vec{v}\cdot\nabla_x f} - \underbrace{\frac{\partial\phi}{\partial x_i}\cdot\frac{\partial f}{\partial v_i}}_{\nabla\phi\cdot\nabla_v f} = 0$$

COLLISIONLESS BOLTZMANN EQUATION $\boxed{\vec{v} \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0}$

DESCRIBES HOW EXACTLY THE SYSTEM THAT IS GUIDED BY NEWTONIAN DYNAMICS AND WHICH CONSERVES PROBABILITY MUST EVOLVE.

POZN. ADVECTIVE DERIVATIVE

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{\alpha} \dot{w}_{\alpha} \frac{\partial f}{\partial w_{\alpha}}$$ (FROM THE CHAIN RULE)

$$\longrightarrow \frac{df}{dt}(\vec{r}, \vec{v}, t) = 0$$ FLOW OF PHASE SPACE DENSITY INCOMPRES.

BOLTZ. EQ. - COMPLICATED OBJECT, IT DEPENDS ON 3 SPATIAL COORD, 3 VELOCITY COORD. AND TIME - 7 COORD FOR EVERY PARTICAL

B7 JEANS EQUATIONS

- COLLISIONLESS BOLTZMANN EQUATION DEPENDS ON 7 VARIABLES $\vec{x}(3)$, $\vec{v}(3)$ AND $t(1)$, SPACE POSSITION - ok, BUT VELOCITY IS DIFICULT TO OBSERVE - SPECTROSCOPICALY ONLY ALONG THE LINE OF SIGHT
- SOLUTION - TAKE VELOCITY MOMENTS ⇒ INTEGRATE COLLISIONLESS BOLTZMANN EQUATION OVER VELOCITY SPACE

$$\int d^3v \frac{\partial}{\partial t} f + \int d^3v \, v_i \frac{\partial f}{\partial x_i} - \int d^3v \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

TERMS DON'T DEPEND ON $\vec{v}$

$$\frac{\partial}{\partial t} \int d^3v \, f + \frac{\partial}{\partial x_i} \int d^3v \, f \cdot v_i - \frac{\partial \phi}{\partial x_i} \cdot \underbrace{\int d^3v \frac{\partial f}{\partial v_i}}_{\to 0 \text{ IF } f \text{ VANISHES AT INFINITY}} = 0$$

LET'S DEFINE TWO THINGS - $\rho = \int d^3v \, f$ ~ MARGINALISATION

DENSITY OF THE STARS (OBJECTS)

DISTRIB. OF STARS

$- \rho \langle v_i \rangle = \int d^3v \, f \cdot v_i$

THEN WE GET

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho \langle v_i \rangle) = 0$$ CONTINUITY EQ.

FROM 6+1 DIMENSIONS WE END UP WITH CONTINUITY EQUATION.

NEXT STEP - APPLY WEIGHTING WITH VELOCITY $\vec{v}$

$$\frac{\partial}{\partial t}\int d^3v\, f\cdot v_j + \int d^3v\, \frac{\partial f}{\partial x_i} v_i v_j - \frac{\partial \phi}{\partial x_i}\int d^3v\, \frac{\partial f}{\partial v_i} v_j = 0$$

LETS LOOK AT THE LAST TERM:

$$\int d^3v\, \frac{\partial f}{\partial v_i}\cdot v_j = -\underbrace{\int d^3v\, f}_{\rho}\,\frac{\partial v_j}{\partial v_i} = -\delta_{ij}\cdot\underbrace{\int d^3v\, f}_{\rho} = -\delta_{ij}\,\rho$$

GAUSS-THEOREM, ASSUMING THAT $\int d^3v\, \frac{\partial}{\partial v_i}(f\cdot v_i) = 0$ AS BEFORE

$$\rightarrow \frac{\partial}{\partial t}(\rho\langle v_j\rangle) + \frac{\partial}{\partial x_i}(\rho\langle v_i v_j\rangle) - \rho\frac{\partial \phi}{\partial x_i} = 0$$

WITH VELOCITY DISPERSION $\rho\langle v_i v_j\rangle = \int d^3v\, f\, v_i v_j$

SUBTRACT $\langle v_j\rangle$

$$\rho\frac{\partial}{\partial t}\langle v_j\rangle - \langle v_j\rangle\cdot\frac{\partial}{\partial x_i}(\rho\langle v_i\rangle) + \frac{\partial}{\partial x_i}(\rho\langle v_i v_j\rangle) = -\rho\frac{\partial\phi}{\partial x_j}$$

LET'S BREAK UP $\langle v_i v_j\rangle = \sigma_{ij}^2 + v_i v_j$

$$\rightarrow \rho\frac{\partial}{\partial t}\langle v_j\rangle + \rho\cdot\langle v_i\rangle\frac{\partial}{\partial x_i}\langle v_j\rangle = -\rho\frac{\partial\phi}{\partial x_j} - \frac{\partial}{\partial x_i}(\rho\,\sigma_{ij}^2)$$

LOOKS LIKE

IN ANALOGY TO THE EULER EQUATION IN FLUID ~~DYNAMICS~~ MECHANICS WITH σ_{ij}^2 AS AN ANISOTROPIC SOURCE OF PREASSURE.

CLOSURE PROBLEM

(0) CONTINUITY, NEED SOLUTION OF $\vec{v}$

(1) MOMENTUM EQUATION NEEDS VELOCITY DISPERSION σ_{ij}

IN GENERAL, THE MOMENT $\langle v^N\rangle$ IS NEEDED FOR $\langle v^{N-1}\rangle$

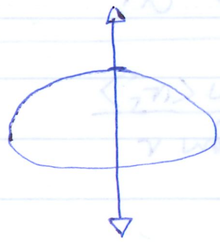
$\rightarrow$ INFINIT LINKED EQUATIONS OF MOMENT.

B8 ANISOTROPIC PREASSURE IN EXAMPLE ELLIPTICAL GALAXIES

VLASTNÍ ~~HODNOTY~~ HODNOTY

UNEQUAL EIGENVALUES OF σ_{ij}^2 CAUSE AN ASPHERICAL SHAPE

$$\sigma^2 \sim \begin{pmatrix} \sigma_x^2 & & 0 \\ & \sigma_y^2 & \\ 0 & & \sigma_z^2 \end{pmatrix} + \text{DIAGONAL}$$



$\to$ IF CONSTANT: $\frac{\partial \phi}{\partial x_i} = -\sigma_i \cdot \delta_{ij} \cdot \frac{\partial \ln \rho}{\partial x_j}$; $\rho \sim \exp\left(-\int dx_i \, \sigma_i\right)$

SPHEROID SYSTEM: ASSUME (1) STEADY STATE

(2) $\langle v \rangle = 0$ SYSTEM DOES NOT MOVE AROUND)

$$\to \frac{d}{dr}\left(\rho \langle v_r^2 \rangle\right) = -\frac{\rho}{r}\left[2 \cdot \langle v_r^2 \rangle - \left(\langle v_\varphi^2 \rangle + \langle v_\theta^2 \rangle\right)\right] = -\rho \frac{d\phi}{dr}$$

INVARIANCE ROTATION $\langle v_\varphi^2 \rangle = \langle v_\theta^2 \rangle$

ANISOTROPY PARAMETERS β $\quad \beta = 1 - \frac{\langle v_\theta^2 \rangle}{\langle v_r^2 \rangle}$ IF $\neq 0$ DESCRIB. ANISOTR.

$$\to \frac{1}{\rho} \frac{d}{dr}\left(\rho \cdot \langle v_r^2 \rangle\right) + \frac{2\beta}{r} \langle v_r^2 \rangle = -\frac{d}{dr}\phi$$

IN THE MILKY WAY $\beta \geq 0$

B9 MASS DETERMINATION OF A SPHEROIDAL SYSTEM

$\langle v_r^2 \rangle$, β AND ρ ARE OBSERVABLE

THROUGH STELLAR DENSITY, WIDTH OF SPECTRAL LINEN +

+ ELIPTICITY

$\frac{d\phi}{dr} = -\frac{GM(r)}{r^2}$ NEWTONIAN GRAVITATIONAL LAW WITH MASS $M(r)$

$M(r) = \int_0^r 4\pi r^2 dr \cdot \rho$ MASS CONTAIN IN RADII SMALLER THAN r

LET'S WRITE IT DIFFERENTLY

$$\frac{GM(r)}{r} = -\langle v_r^2 \rangle \cdot \left(\frac{d \ln \rho}{d \ln r} + \frac{d \ln \langle v_r^2 \rangle}{d \ln r} + 2\beta\right)$$

USING $\frac{1}{\rho} \underbrace{\frac{d}{dr}\left(\rho \cdot \langle v_r^2 \rangle\right)}_{= \rho \cdot \frac{d \langle v_r^2 \rangle}{dr} + \langle v_r^2 \rangle \frac{d\rho}{dr}} + \frac{2\beta}{r} \langle v_r^2 \rangle = -\frac{d\phi}{dr} = -\frac{GM(r)}{r^2} \quad \Big/ \cdot \frac{r}{\langle v_r^2 \rangle}$

$$\cancel{\rho \cdot \frac{d\langle v_r^2\rangle}{dr} + \langle v_r^2\rangle \frac{d\rho}{dr}}$$

$$\underbrace{\frac{r}{\langle v_r^2\rangle}\frac{d}{dr}\langle v_r^2\rangle}_{=\frac{d\ln\langle v_r^2\rangle}{d\ln r}} + \underbrace{\frac{r}{\rho}\frac{d\rho}{dr}}_{=\frac{d\ln\rho}{d\ln r}} + 2\beta = -\frac{GM}{r}\frac{1}{\langle v_r^2\rangle}$$

SPHERICAL CASE: $\beta = 0$

$$\rightarrow GM(r) = -r\cdot\langle v_r^2\rangle\cdot\left(\frac{d\ln\langle v_r^2\rangle}{d\ln r} + \frac{d\ln\rho}{d\ln r}\right)$$

THIS EQUATION NOW ALLOWS US TO DETERMINE MASSES, WE NEED TO MEASURE VELOCITY DISPERSION, THEN WE NEED TO MEASURE HOW FAST THE VELOCITY DISPERSION CHANGES → WE NEED TO MEASURE HOW THE DENSITY OF STARS CHANGES WITH RADIUS.

B10 VIRIAL RELATIONS

- STABLE ABOUT THE SYSTEM IF YOU ALREADY KNOW THE SOLUTION.

- WE USE MOMENTUM EQUATION - FIRST VELOCITY MOMENT IN PHASE DISTRIBUTION

$$\frac{\partial}{\partial t}(\rho\langle v_i\rangle) + \frac{\partial}{\partial x_j}(\rho\langle v_i v_j\rangle) = -\rho\frac{\partial\phi}{\partial x_i} \qquad /\cdot x_k \int d^3r$$

$$\int d^3r\, x_k \frac{\partial}{\partial t}(\rho\langle v_i\rangle) = -\underbrace{\int d^3r\, x_k\frac{\partial}{\partial x_j}(\rho\langle v_i v_j\rangle)} - \int d^3r\, x_k\rho\frac{\partial\phi}{\partial x_i}$$

- FIRST, LET'S LOOK AT THIS TERM

FIRST TERM - KINETIC ENERGY TENSOR ($\rho \cdot$ VELOCITY^2 $\sim E_k$ + DIVERGENCE -- "TENSOR")

$$\int d^3r\, x_k \frac{\partial}{\partial x_j}\left(\rho \cdot \langle v_i v_j \rangle\right) = \overbrace{\int d^3r \frac{\partial}{\partial x_j}\left(x_k \rho \langle v_i v_j\rangle\right)}^{=0 \text{ FROM GAUSS THEOREM.}} - \int d^3r\, \rho \langle v_i v_j\rangle \cdot \underbrace{\frac{\partial x_k}{\partial x_j}}_{\delta_{kj}} =$$

$$= -K_{jk} = \int d^3r\, \rho \langle v_j \cdot v_k \rangle$$

K_{jk} - KINETIC ENERGY TENSOR

CHANDRASEKHAR POTENTIAL ENERGY

$$W_{jk} = -\int d^3r\, x_k \cdot \frac{\partial \phi}{\partial x_j}$$

NOW LET'S SPLIT UP KINETIC TENSOR INTO AVERAGED AND FLUCTUATING PART.

$$K_{jk} = \frac{1}{2}\Pi_{jk} + T_{jk}$$

AND THESE TWO CONTRIBUTIONS ARE EQUAL:

$$T_{jk} = \frac{1}{2}\int d^3r\, \rho \langle v_j\rangle \langle v_k\rangle \quad \& \quad \Pi_{jk} = \int d^3r \cdot \rho \cdot \langle v_j v_k\rangle$$

$$\frac{1}{2}\frac{d}{dt}\int d^3r\, \rho \cdot \left(x_k \langle v_j\rangle + x_j \langle v_k\rangle\right) = 2K_{jk} + W_{jk}$$

THIS IS VALID BASED ON SYMMETRISATION FROM KINETIC ENERGY TENSOR.

INERTIA TENSOR $\quad I_{jk} = \int d^3r\, \rho\, r_j r_k$

- ~~...~~ SECOND MOMENT OF MATTER DISTRIBUTION

TIME DERIVATIVE:

$$\frac{d}{dt} I_{jk} = \int d^3r \frac{\partial \rho}{\partial t} \cdot x_j x_k \overset{\text{SUBSTITUTING CONTIN. EQU.}}{=} -\int d^3r \frac{\partial(\rho \langle v_i \rangle)}{\partial x_i} x_j x_k =$$

$$= \underbrace{-\int d^3r \frac{\partial}{\partial x_i}(\rho \langle v_i\rangle x_j x_k)}_{=0 \text{ GAUSS THEOR.}} + \int d^3r\, \rho \langle v_i\rangle \underbrace{\frac{\partial}{\partial x_i}(x_j x_k)}_{\delta_{ij} x_k + x_j \delta_{ik}} =$$

$$= \int d^3r\, \rho \cdot \langle v_i\rangle (x_k \delta_{ij} + x_j \delta_{ik}) = \int d^3r\, \rho (\langle v_j\rangle x_k + \langle v_k\rangle \cdot x_j) = \frac{d}{dt} I_{jk}$$

NOW COMBINE WITH $\frac{1}{2}\frac{d}{dt}\int d^3r\,(\rho x_k \langle v_j\rangle + x_j \langle v_k\rangle) = 2K_{jk} + W_{jk}$

$$\rightarrow \frac{1}{2}\frac{d^2}{dt^2} I_{jk} = 2K_{jk} + W_{jk} = 2T_{jk} + \Pi_{jk} + W_{jk}$$

STATIONARY SITUATION: $\frac{d}{dt} I = 0 \rightarrow 2K_{jk} + W_{jk} = 0$

TENSORS

TENSOR VIRIAL THEOREM

B11 SCALAR VIRIAL THEOREM

- TRACE OF TENSORS, THAT'S HOW WE DO IT. THAT WILL GIVE US SCALAR STATEMENT.

$$\mathrm{tr}(K_{jk}) = \int d^3r \cdot \rho \cdot \langle v_j v_j\rangle = \frac{1}{2}\int d^3r\, \rho \langle v^2\rangle = \langle v_j v_k\rangle \delta_{jk}$$

SCALAR PRODUCT OF KIN. ENERGY

$\mathrm{tr}(W_{jk}) = \frac{1}{2} \cdot \int d^3r\, \rho\, \phi$ POTENTIAL ENERGY

$$2K + W = 0$$

$$W_{ij} = -\int d^3r\, \rho \cdot x_i \frac{\partial}{\partial x_j}\phi \qquad \phi = -G\int d^3r' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$W_{ij} = G\int d^3r \int d^3r'\, \rho(\vec{x}) \cdot \rho(\vec{x}') \cdot x_i \frac{x_j - x_j'}{|\vec{x}-\vec{x}'|^3}$$

GRADIENT IN j-DIRECTION

$$\frac{\partial}{\partial x_j} \frac{1}{|\vec{x}-\vec{x}'|}$$

NOW INTERCHANGING $x \rightleftarrows x'$ AND THE ORDER INTEGRATION AS WELL:

$$W_{ij} = -G \int d^3r' \int d^3r \, \rho(\vec{x}) \cdot \rho(\vec{x}') \cdot x'_j \cdot \frac{|\vec{x}'_i - \vec{x}_i|}{|\vec{x} - \vec{x}'|^3} \Rightarrow$$

$$tr(W_{ij}) = -\frac{G}{2} \int d^3r \int d^3r' \cdot \rho(\vec{x}) \cdot \rho(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|} =$$

CONTRIBUTION TO POTENTIAL TIME ρ

$$= -\frac{G}{2} \cdot \int d^3r \, \rho(\vec{x}) \cdot \int d^3r \, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} = \frac{1}{2} \int d^3r \, \rho(\vec{x}) \cdot \phi(\vec{x}) \equiv W$$

B13 VIRIAL RELATIONSHIPS

→ WHAT EVER YOU DO SYSTEM, PLEASE TAKE CARE THAT SUM OF YOUR ENERGY IS CONSERVE, NOW THE VIRIAL RELATIONSHIP MAKE A VERY MUCH CONSTRAINING STATEMENT ABOUT THIS, THEY SAY: WELL IT IS TYPICAL FOR SYSTEMS ~~SPEN~~ TO KEEP THEIR ENERGY IN ONE FORM ON AVEREGE IN TIME THERE IS AS MUCH AS KINETIC ENERGY AS MUCH THERE IS POTENTIAL.

- **ESTIMATION PURPOSES**, USING IT IN PHYSICAL ~~SYSTEMS~~ SYSTEMS IS COMPLICATED BECAUSE WE HAVE TO FIND OUT WHETHER $\frac{1}{2}\frac{d^2}{dt^2}I$ IS ZERO OR NOT. WE CAN SAY OKAY LET HAND SIDE OF

$$\boxed{\frac{1}{2}\frac{d^2}{dt^2}I = 2k_{jk} + W_{jk}}$$

THIS EQUATION IS VERY SMALL AND WE CAN ESTIMATE THE KINETIC AND POTENTIAL ENERGY.

WE WILL SAY THAT THE POTENTIAL ENERGY IS

$$W \sim \frac{GM^2}{R} \longrightarrow R \sim \frac{GM^2}{W}$$

($2k + W = 0$ FOR STATIONARY, ISOLATED OBJECTS)

$W \cong \frac{k}{2} \qquad k \sim M \langle v^2 \rangle \qquad \boxed{R = \frac{GM}{\langle v^2 \rangle}}$

WE HAVE NOW RELATION BETWEN SIZE OF THE SYSTEM AND MASS AND POTENTIAL ENERGY. IF THERE IS VIRIAL EQUILIBRIUM (STABLE SYSTEM I GUESS) AND "COULOMB TYPE" OF FORCE ACTING BETWEEN PARTICLES.

SIZE IS FIXED BETWEEN RATIO $\frac{M}{\langle v^2 \rangle}$ IF YOU FORCE THE SYSTEM TO HAVE ANOTHER VELOCITY DISPERSION (FOR EXAMPLE I PUT TOGETHER SOME STARS OF SAME SIZE & MASS AND GIVE THEM TO LARGE VELOCITY DISPERSION OF COURSE THE OBJET WILL EXPAND - IT'S CONFINED INTO SMALL SPACE WITH TO CONSERVATIVE INITIAL CONDITIONS.

R - TOO LITTLE, THUS $\frac{1}{2}\frac{d^2}{dt^2} I = 0$ NOT FULLFILLED, SYSTEM REACTS BY CHANGING IT'S INERTIA TENSOR.

SYSTEM ALWAYS TRIES TO FIND EQUILIBRIUM.

IF $\frac{1}{2}\frac{d^2}{dt^2} I \neq 0$ SYSTEM WILL EVOLVE IN TIME UNTIL IT FULLFILL THAT IT'S EQUAL 0.

ORBITS PLANETS AROUND THE SUN EVENTUALLY FULLFILL THE VIRIAL THEOREM.

B.14 MASS TO LIGHT RATIO (EVIDENCE FOR DARK MATTER)

- LET'S START WITH SPHERICAL SYSTEM, ISOLATED, VIRIALISED (CLUSTER OF GALAXIES)
- ISOLATED - NO PREASSURE TERM (FOR CLUSTER OF GALAXIE - OKAY (ISH))
- VIRIALISED - WE CAN DEFINE AVERAGES FOR KINETIC & POTENTIAL ENERGY.

LET'S LINK THESE THINGS TO OBSERVATIONAL PROPERTIES...

LUMINOSITY FUNCTION (DISTRIBUTION) $\mu = \frac{\rho(\vec{x})}{\Upsilon}$

Υ – MASS TO LIGHT RATIO $\frac{\text{SOLAR MASS}}{\text{SOLAR LUMINOSITY}}$

ρ – MAP WHERE THE MATTER IS (MATTER DENSITY)

μ – MAP WHERE THE LIGHT IS

$\rho(\vec{x}) = \Upsilon \cdot \mu$

x-COMPONENTS OF KINETIC ENERGY:

$$k_{xx} = \int d^3x \, \frac{1}{2} \cdot \rho(\vec{x}) \cdot \underbrace{\langle v_x^2 \rangle}_{\text{TRACE OF KIN ENERGY}} = \frac{\Upsilon}{2} \cdot \int d^3x \, \mu(\vec{x}) \cdot \langle v_x^2 \rangle \quad (\text{FOR CONSTANT } \Upsilon)$$

FOR SPHERICAL SYSTEM $k_{xx} + k_{yy} + k_{zz} = 3k_{xx}$

THIS WE CAN OBSERVE:

- WE CANNOT OBSERVE THE DISTRIBUTION OF LUMINOSITY (IN 3D)
- WE CAN TALK ABOUT SURFACE BRIGHTNESS PROFILE $\underline{\Sigma(R)}$
- R – SIZE OF THE OBJECT,
- WE CANNOT MEASURE MOTION PERPENDICULAR TO THE LINE OF SIGHT, WE CAN SE MOTION IN THE LINE OF SIGHT (RELATED TO THE DOPPLER EFFECT) $\sigma_n^2(R)$

KINETIC ENERGY

$$k = \frac{3}{2} \Upsilon \int d\varphi \int R \, dR \, \Sigma(R) \cdot \sigma_{\parallel}^2(R) = 3 \cdot \pi \cdot \Upsilon \int_0^{\infty} R \, dR \cdot \underbrace{\Sigma \sigma_n^2}_{\text{OBSERVABLE}} = J \cdot \Upsilon$$

POTENTIAL ENERGY

$$w = -\frac{G}{2} \int dr \, \frac{M(r)^2}{r^2} \quad \Big| \quad \text{FOR SPHERICAL SYSTEMS } M = 4\pi \int r^2 \, dr \, \rho(r)$$

$$\rho(r) = -\frac{\Upsilon}{\pi} \cdot \int_0^{\infty} dR \, \frac{1}{\sqrt{R^2 - r^2}} \, \frac{d\Sigma}{dR} \qquad \text{ABEL INTEGRAL}$$

$$\rightarrow w = -G\Upsilon^2 \underbrace{\int_0^{\infty} \frac{dr}{r^2} \Big(\underbrace{\int_0^{r} r'^2 \, dr' \underbrace{\int_{r'}^{\infty} dR \, \frac{1}{\sqrt{R^2 - r^2}} \, \frac{d\Sigma}{dR}}_{\text{THIS IS DENSITY}}}_{\text{TOTAL MATTER}} \Big)^2}_{\text{TOTAL POTENTIAL ENERGY} \sim \frac{M^2}{r^2}} = \Upsilon^2 \cdot \tilde{J}$$

FROM VIRIAL THEOREM WE KNOW $2K+W=0 \rightarrow -\frac{2K}{W}=1$

$$K=\Upsilon\cdot J \; ; \; W=\Upsilon^2\cdot\tilde{J} \longrightarrow$$

$$-\frac{2\Upsilon J}{\Upsilon^2\tilde{J}}=1 \Rightarrow -\frac{2J}{\Upsilon\tilde{J}}=1$$

J IS LINKED TO OBSERVATION VIA $\underline{\Sigma(R)\,;\,\sigma_{\parallel}^2}$ ← THIS WE NEED

$$\boxed{\Upsilon=-\frac{2J}{\tilde{J}}}$$

IN GALAXY CLUSTERS: Υ ~ FEW HUNDRED (~~TO~~ THOUSANDS) $\frac{M_\odot}{L_\odot}$

B15 JEANS THEOREM

- CLASICAL SYSTEM STORES ENERGY IN A FORM OF KINETIC OR POTENTIAL ENERGY. LAGRANGE FUNCTION DOES NOT DEPEND ON TIME IF ~~AND~~ TOTAL ENERGY IS CONSERVE.

- NOW WE WILL LOOK AT SYSTEMS THAT STABLEZE IT SELF. JEANS EQUATION & BOLTZMAN EQUATION DEISCARD ANY TIME EVOLUTION OF THE SYSTEM.
- IF THE SYSTEM CHANGES IT'S MOMENT OF INERTIA (IN GENERAL), THEREFORE CHANGES ITS SHAPE, THE RATE OF CHANGES IN SYSTEM IS RELATED TO MEAN KINETIC & POTENTIAL ENERGY.
- SYSTEM KEEPS EVOLVING IF ITS SHAPE IS CHANGING.

100 MIL YEAR AROUND THE GALAXY

LET'S ASSUME THAT f IS A FUNCTION OF THE INTEGRALS OF MOTION $f(I_n)$: $\longrightarrow \frac{df}{dt}=\sum \underbrace{\frac{dI_n}{dt}}_{0}\frac{\partial f}{\partial I_n}=0$

BECAUSE EACH FACTOR IS CONSTANT IN TIME

CONSTANTS OF MOTION I_n, PHASE SPACE DISTRIB. FUNCTION $f(I_n)$

B16 SPHERICAL SYSTEMS

– NO NET ~~A~~ ANGULAR MOMENTUM

SO SPACE DISTR. FUNCTION $f = f(E)$ (NO NET ANGUL. MOM. $\vec{L}$)

$$E = \phi(\vec{r}) + \frac{1}{2}\vec{v}^2 = \phi + \frac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2)$$ SPHER. COORDINATES.

IF f DEPENDS ONLY ON KINETIC & POTENTIAL ENERGY THAN WE CAN COMPUTE VELOCITY MOMENT

$$\rho\langle v_r^2\rangle = \int d^3v \cdot v_r^2 \cdot f\left(\phi + \tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2)\right)$$

$$\rho\langle v_\theta^2\rangle = \int d^3v \cdot v_\theta^2 \cdot f\left(\phi + \tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2)\right)$$

$$\rho\langle v_\varphi^2\rangle = \int d^3v \cdot v_\varphi \cdot f\left(\phi + \tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2)\right)$$

$\langle v_r^2\rangle = \langle v_\theta^2\rangle = \langle v_\varphi^2\rangle$

SPHERICAL SYSTEM, ISOTROPIC

ALSO:

NO NET DISPLACEMENT OF ROTATION $\langle v_r\rangle = \langle v_\theta\rangle = \langle v_\varphi\rangle$

B17 SELF-CONSISTENT, SELF-GRAVITING SYSTEMS

→ GRAVITATIONAL POTENTIAL IS PROVIDED BY THE SYSTEM IT SELF (NOT EXTERNALY)

$\psi = -\phi + \phi_0$
$\mathcal{E} = -E + \phi_0$ RELATIVE POTENTIALS/ENERGIES

$f > 0$ FOR $E > 0$
$f = 0$ FOR $\mathcal{E} \leq 0$

CONSTRUCTION OF A SELF CONSISTENT MODEL $\rho = \int d^3v\, f$ →

→ $\Delta\phi = 4\pi G \cdot \rho$ PROVIDES THE GRAVITY ITSELF ~ 'SELF-GRAVITING SYS.'

(POISSON EQ.)

COMES FROM $\rho = \int d^3v\, f$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = -16\pi^2 G \cdot \int_0^\psi d\mathcal{E}\, f(\mathcal{E}) \cdot \sqrt{2(\psi - \mathcal{E})}$$

$d^3v = 4\pi v^2 dv \; ; \; v^2 = 2(\psi - \mathcal{E})$

$$\frac{dr}{d\varepsilon} = \frac{-1}{\sqrt{2(\psi-\varepsilon)}}$$

$$\rightarrow d^3v = 4\pi v^2 dv = 4\pi v^2 \left|\frac{dv}{d\varepsilon}\right| d\varepsilon = 4\pi \cdot 2 \cdot (\psi - \varepsilon) \cdot \frac{1}{\sqrt{2(\psi-\varepsilon)}} d\varepsilon$$

ALLOWS TO COMPUTE ψ FROM f OR f FROM ψ

B.4 $\rho \rightarrow f$ FOR SINGULAR ISOTHERMAL SPHERE,
ASSUME SPHERICAL VELOCITY DISTRIBUTION

$$f(\varepsilon) = \frac{\bar{\rho}}{\sqrt{2\pi\sigma_0^2}} \cdot e^{\left(\frac{(\psi - \frac{1}{2}\vec{v}^2)}{\sigma^2}\right)}$$

(LOOKS A BIT LIKE THE MAXWELL DISTRIBUTION, WHERE σ_0^2 CORESPOND TO TEMPERATURE)

$$\rho(\varepsilon) = 4\pi \cdot \int_0^{\sqrt{2\psi}} v^2 dv\, f(\varepsilon) = 4\pi \int_0^{\psi} d\varepsilon\, f(\varepsilon) \cdot \sqrt{2(\psi-\varepsilon)} = \bar{\rho} \exp\left(\frac{\psi}{\sigma_0^2}\right)$$

$\longrightarrow$ GOING BACK TO THAT POISONIC EQUATION

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = -4\pi G \cdot \bar{\rho} \exp\left(\frac{\psi}{\sigma_0^2}\right)$$

$$\frac{d\psi}{dr} = -\frac{4\pi G \bar{\rho}}{r^2} \cdot \int_0^r r^2 dr\, e^{\left(\frac{\psi}{\sigma_0^2}\right)}$$

SOLVED BY $\psi(r) = -2\sigma_0^2 \ln r$

$$\rho(r) = \frac{\sigma_0^2}{2\pi G r^2}$$

SINGULAR ISOTHERMAL SPHERE

$\rho \rightarrow 0$ AT $r=0$ $\sigma_0^2 = $ KONST.

LET'S LOOK AT JEANS EQUATION

$$\frac{1}{\rho}\frac{d}{dr}(\rho\sigma^2) = \frac{d\psi}{dr} \longrightarrow \sigma^2 = \sigma_0^2$$

B19 $f \to \rho$ EDDINGTONS FORMULA

If Ψ IS A MONOTONIC FUNCTION OF r (NOT NECESSARILY TIME)
$\to \rho$ CAN BE UNAMBIGUOSLY IN TERM OF Ψ ... $\rho \to \rho(\Psi)$
$\rho(\Psi(r))$ & ALSO $\rho(r)$

$$\rho(\Psi) = \int d^3r \, f = 4\pi \int_0^{\Psi} d\mathcal{E} \, f(\mathcal{E}) \cdot \sqrt{2(\Psi - \mathcal{E})} \quad \Big| \frac{d}{d\Psi}$$

FOR MONOTONIC POTENTIAL $\Psi(r)$

$$\to \frac{1}{\sqrt{8}\,\pi} \frac{d\rho}{d\Psi} = \int_0^{\Psi} d\mathcal{E} \frac{f(\mathcal{E})}{\sqrt{\Psi - \mathcal{E}}} \quad \text{BY TAKING } \frac{d}{d\Psi}$$

IT'S SOLVED BY: $f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \cdot \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi \frac{d\rho}{d\Psi} \frac{1}{\sqrt{\mathcal{E} - \Psi}} =$

$$= \frac{1}{\sqrt{8}\,\pi^2} \cdot \left(\int_0^{\Psi} d\Psi \frac{d^2\rho}{d\Psi^2} \cdot \frac{1}{\sqrt{\mathcal{E} - \Psi}} + \frac{1}{\sqrt{\mathcal{E}}} \frac{d\rho}{d\Psi}\Big|_{\Psi=0} \right)$$

DEFINES $f(\mathcal{E})$ FROM THE DENSITY $\rho(\Psi)$, BE CAREFUL: $f(\mathcal{E}) \geq 0$ REQUIRES THAT $\int_0^{\mathcal{E}} d\Psi \frac{d\rho}{d\Psi} \frac{1}{\sqrt{\mathcal{E} - \Psi}}$ IS INCREASING WITH $\mathcal{E}$, WHICH PUTS A ~~CONSTANT~~ ~~ON~~ CONSTRAIN ON $\frac{d\rho}{d\Psi}$

B20 RELAXATION

- FLUID MECHANICAL SYSTEMS CAN FIND EQUILIBRIUM CONFIGURATIONS THROUGH THE EQUATION OF STATE AND BY ADJUSTING THE TEMPERATURE BY VISCOUS DISSIPATION.
- MICROSCOPIC ORIGIN OF PREASSURE AND VISCOSITY ARE COLLISIONS BETWEEN "PARTICLES", BUT THESE COLLISIONS ARE EXTREMLY RARE.
- IMAGINE A SYSTEM OF SIZE R WITH N STARS (BODIES) WHICH HAVE SIZE r

(PROJECTED)

CROSS SECTION: $\sigma = 4\pi R^2$ = GEOMETRIC SURFACE (SIZE OF EACH STAR)

MEAN FREE PATH:

$$\lambda = \frac{1}{n \cdot \sigma} \quad ; \quad n = \frac{3N}{4\pi R^3} \text{ NUMBER DENSITY}$$

$$\rightarrow \frac{\lambda}{R} = \frac{4\pi R^3}{3N 4\pi r^2 R} \simeq \left(\frac{R}{r}\right)^2 \cdot \frac{1}{N}$$

AS THE NUMBER OF STARS GO UP ↑ THE MEAN FREE PATH DECRESES ↓.

CROSSING TIME SCALE: $t_{cross} = \frac{R}{v}$, COLLISION TIME SCALE $t_{coll} = \frac{\lambda}{v}$

$$\frac{t_{coll}}{t_{cross}} = \left(\frac{R}{r}\right)^2 \frac{1}{N}$$

LOOK AT THE MILKY WAY

$R = 10 \text{ kpc} \simeq 10^{21} \text{ m}$; $v \simeq 100 \frac{km}{s} = 10^5 \text{ m s}^{-1}$; $N = 10^{10}$ STARS

SIZE OF A STAR $r \simeq 10^5 \text{ km} \simeq 10^8 \text{ m}$

$$\frac{\lambda}{R} = \left(\frac{10^{21}}{10^8}\right) \frac{1}{10^{10}} = 10^{16} \qquad \frac{t_{coll}}{t_{cross}} = 10^{16}$$

$$t_{cross} \simeq 10^{16} \text{ s} \quad ; \quad t_{coll} = 10^{32} \text{ s} = 10^{15} \frac{1}{H_0}$$

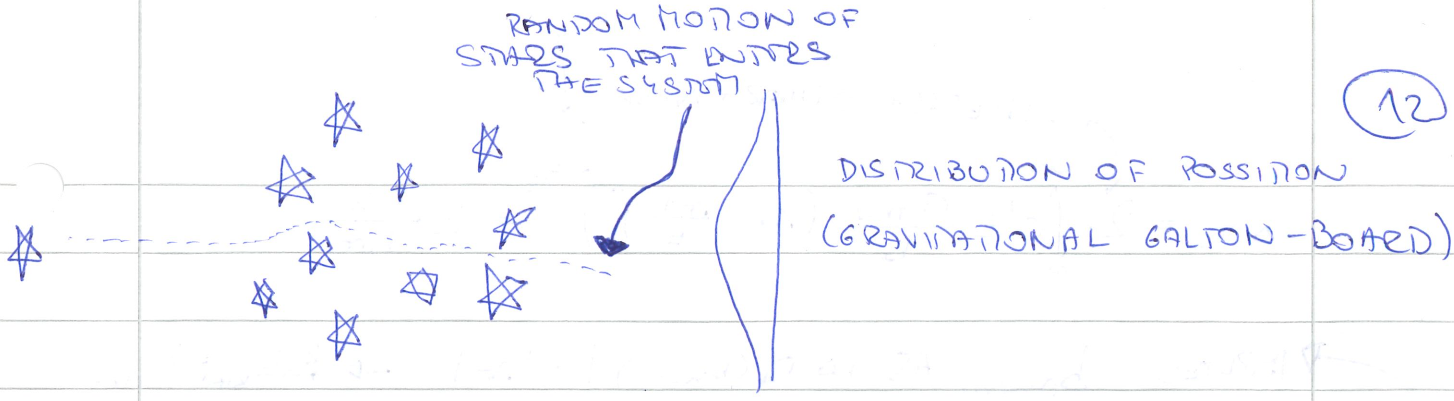
TIME BEFORE STARS MEET EACH OTHER

H_0 - HUBBLE CONSTANT $H_0 = \frac{100 \frac{km}{s}}{Mpc} \Rightarrow \frac{1}{H} = 10^{27} \text{ s} \sim$ AGE OF UNIVERSE

STARS PRETTY MUCH NEVER COLLIDE...

B2.1 RELAXATION TIME

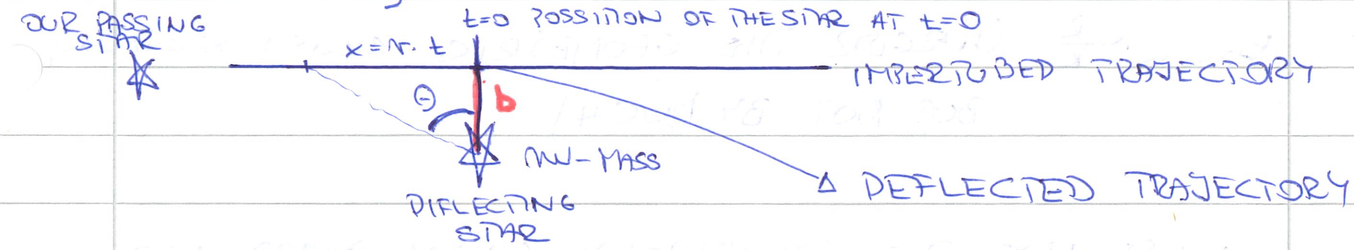
- WE HAVE A COLLECTION OF STARS AND ONE STAR THAT ENTERING IN THIS GROUP.



→ ~~STAR~~ THIS STAR WILL NEVER HIT OTHER STARS BUT WILL BE GRAVITATIONALY INFLICTED BY THEM

- LET'S MONITOR THIS VELOCITY PERPENDICULAR TO THE INITIAL VELOCITY.

- ONE DIFLECTING STAR - SINGLE KEPLERIAN ORBIT OF A PASSING STAR



b ~ IMPACT PARAMETER, $x = v \cdot t$ OF THE PASSING STAR, SET $t=0$ AT $x=0$

FROM GEOMETRY WE GET $\cos\Theta = \dfrac{b}{\sqrt{x^2+b^2}} = \left(\dfrac{x^2+b^2}{b^2}\right)^{-1/2} =$

$$= \left(\frac{x^2}{b^2} + 1\right)^{-1/2} = \left(1 + \left(\frac{v \cdot t}{b}\right)^2\right)^{-1/2}$$

GRAVITATIONAL FORCE $\quad F_\perp = \dfrac{G \cdot m^2}{x^2+b^2} \cdot \cos\Theta = \dfrac{G m^2}{b^2} \cdot \left(1 + \left(\dfrac{vt}{b}\right)^2\right)^{-3/2}$

- TOTAL CHANGE IN VELOCITY ⊥ PERPENDICULAR TO THE INITIAL DIRECTION, WE HAVE TO INTEGRATE ALONG UNPERTURBED TRAJECTORY → BORN APPROXIMATION

BORN CONDITION $|\Delta v_\perp| \ll |v|$, WITH $\dfrac{d}{dt} \Delta v_\perp = \dfrac{F_\perp}{m}$

CHANGE IN THE ⊥ VELOCITY HAS TO BE SMALL IN COMPARISON TO THE ORIGINAL VELOCITY.

SYMETRY OTHER+WISE $\int_{-\infty}^{\infty}$

$$\Delta v_+ = 2 \cdot \int_0^\infty dt \, \frac{Gm}{b} \left(1 + \left(\frac{vt}{b}\right)^2\right)^{-3/2} = 2 \frac{Gm}{bv}$$

IMPLIES b_{min} AS $|\Delta v_+ (b_{min})| = |v| \rightarrow \Delta E_{kin} = E_{kin}$

$$b_{min} = \frac{2Gm}{v^2} \sim \frac{R}{N} ; \text{ FOR THE MILKY WAY}$$

TOTAL ACCELERATION PERPENDICULAR INCREASES AS b BECOMES SMALLER.

b_{min} ~ FEW TEN SOLAR RADII (FOR MILKY WAY)

$b_{min} \sim \frac{R}{N}$ (LARGER THE GEOMETRIC CROSS SECTION, BUT NOT BY MUCH)

WE USE NEXT THE GRAVITATIONAL GALTON BOARD, STAR IN EACH ENCOUNTER GET SOME ACCELERATION PERPENDICULAR TO THE INITIAL VELOCITY AND DIFLECTIONS ARE UNCORELATED.

$n(<b)$ ~ NUMBER OF PASSES WITH IMPACT PARAMETERS $<b$

$$n(<b) = N \cdot \frac{\pi \cdot b^2}{\pi \cdot R^2} = N \left(\frac{b}{R}\right)^2 \text{ FROM GEOMETRY}$$

NUMBER DENSITY OF TRAJECTORIES LOWER THAN b.

CUMMULATIVE DISTRIBUTION (GO TO DIFFERENTIAL DISTRIBUTION BY TAKING THE DERIVATIVE

$$n(b) = \frac{d}{db} n(<b) \qquad n(b)\,db = \frac{2Nb}{R^2} db$$

WITH BOUNDARY b_{min}

$$n(<b) = \int_{b_{min}}^{R} db \, n(b)$$

THAN WE CAN WRITE DOWN

$$\Delta v_\perp^2(b) = \underbrace{\left(\frac{2Gm}{bv}\right)^2}_{\Delta v_\perp^2} \cdot \underbrace{\frac{2Nb}{R^2}}_{\text{PROBABILITY OF AN ENCOUNTER AT } b} = 8N \cdot \left(\frac{Gm}{Rv}\right)^2 \cdot \frac{1}{b} \Rightarrow$$

$$\Rightarrow \Delta v_\perp^2 \int_{b_{min}}^{R} db\, \Delta v_\perp^2(b) = 8N \left(\frac{Gm}{R\cdot v}\right)^2 \ln\left(\frac{R}{b_{min}}\right)$$

$\ln\left(\frac{R}{b_{min}}\right)$ ~ IT'S CALLED COULOMB-LOGARITHM $\simeq \ln N$

$$\Delta v_\perp^2 = 8 \cdot \left(\frac{GNm}{R}\right)^2 \cdot \frac{1}{v^2} \cdot \frac{\ln(N)}{N} = 8\,\frac{\ln(N)}{N}$$

$\frac{\ln(N)}{N}$ → DIVERGES SLOWLY WITH $N \to \infty$

$\frac{1}{v^2}$ → $v^2 \simeq \frac{GM}{R}$

→ IT NEEDS $\frac{N}{8\ln N}$ CROSSINGS TO CHANGE $\Delta v_\perp^2$ OF ORDER v^2

B22 TYPICAL TIME SCALES IN STELLAR SYSTEMS

① HUBBLE TIME SCALE

$t_H = \frac{1}{H_0} = 10^{17}\,s$ AGE OF THE UNIVERSE

$\frac{c}{H_0} = c \cdot t_H$ SIZE OF THE OBSERVABLE UNIVERSE

② HOW FAST GALAXY FORM (FORMATION TIME SCALE)

$t_f = \frac{M}{\dot{M}} \sim t_H$ FOR A GALAXY

③ CROSSING TIME SCALE $t_{cross} = \frac{R}{v}$

④ COLLISION TIME SCALE $t_{coll} = \left(\frac{R}{r}\right)^2 \cdot \frac{1}{N} \cdot t_{cross}$

(DIRECT COLLISIONS)

(5) RELAXATION TIME SCALES $t_{RELAX} = \frac{N}{f \cdot \ln N} \cdot t_{CROSS}$

(FOR GRAVITATIONAL DEFLECTION)

(6) INTERACTION TIME SCALE

$$t_{SHORT} = N \cdot t_{CROSS}$$

(SHORT RANGE INDIVIDUAL INTERACTIONS)

FOR GALAXIES $t_{CROSS} \ll t_{FORM} \sim t_H \ll t_{RELAX} \ll t_{SHORT} \ll t_{COLL}$

$\hookrightarrow$ OF AGE OF UNIVERSE

TYPICAL NUMBERS

	MASS	RADIUS	VELOCITY	N	t_{CROSS}	t_{RELAX}
GALAXY	$10^{10} M_\odot$	10 kpc	100 km s^{-1}	10^{9}	10^{8} yr	10^{15} yr
CDM HALO	$10^{12} M_\odot$	100 kpc	200 km s^{-1}	10^{50} PART	10^{9} yr	$\sim 10^{60}$ yr
GALAXY CLUSTER	$10^{15} M_\odot$	1 Mpc	1000 km s^{-1}	10^{3} GALAX	10^{9} yr	10^{10} yr
GLOBULAR CLUSTER	$10^{4} M_\odot$	10 pc	FEW km s^{-1}	10^{4} STARS	10^{6} yr	10^{8} yr

B23 FINAL STATES OF SELF-GRAVITATING SYSTEMS

- RELAXATION MECHANISM IS A WAY HOW SYSTEM SOMEHOW FINDS A STABLE STATE.
- COLLISIONAL SYSTEMS THERMALISE, AND REACH A STATE OF MAXIMUM ENTROPY.

$$S = -\int d^3x \int d^3v \, f \cdot \ln f$$

GIBBS-ENTROPY - IS THE MAXWELL ~~GAUSS~~ BOLTZMAN DISTRIBUTION

$\uparrow$ MAXIMIZED

S - IS MAXIMISED BY THE SINGULAR ISOTHERMAL SPHERE, WITH INFINITE DENSITY/MASS, WHICH IS NOT A PHYSICAL SOLUTION.

$U = TS - pV - \mu N + \underbrace{\rho \phi}_{\text{DESCRIBES GRAVITY IN THE SYSTEM}}$ POISON EQUATION $\int d^3 r' \frac{\rho(r')}{|\vec{r} - \vec{r}'|}$

- NO CLEAR SEPARATION BETWEEN INTENSIVE AND EXTENSIVE VARIATIONS.

B24 HEAT CAPACITY OF SELF-GRAVITATING SYSTEMS → NEGATIVE

- SELF-GRAVITATING SYSTEMS HAVE NEGATIVE HEAT CAPACITY

$\frac{m}{2} \langle \vec{v}^2 \rangle = \frac{3}{2} k_B \cdot T$ FOR A COLLISIONAL SYSTEM (EQUIPARTITION OF ENERGY)

MEAN TEMPERATURE $\langle T \rangle = \frac{1}{M} \int d^3x \cdot \rho(\vec{x}) \cdot T(\vec{x})$; $M = \int d^3x \, \rho(\vec{x})$

- HOW TO RELATE KINETIC ENERGY WITH THE TOTAL MASS OF THE SYSTEM

$K = \frac{3}{2} N \cdot k_B \langle T \rangle$

N: NUMBER OF PARTICLES

- NOW LET'S USE VIRIAL LAW

$2K + W = 0$; $E = K + W = K - 2K = -K \Rightarrow E = -\frac{3}{2} N \cdot k_B \langle T \rangle$

- HEAT CAPACITY

$C = \frac{d}{d\langle T \rangle} E = -\frac{3}{2} k_B \cdot N$ NEGATIVE QUANTITY

WHEN WE PUT ENERGY IN A SELF GRAVITATING SYSTEM, THE SYSTEM WILL GET COOL DOWN. OR THE OTHE WAY AROUND, WE COULD TAKE SOME ENERGY FROM THE SYSTEM AND IT WILL HEAT UP. GLOBULAR CLUSTERS EVAPORATE STARS AND THEY WILL GET HOTTER IN THE PROCESS.

- IN THE END ONLY TWO CLOSE BINARIES ARE LEFT.
- BUT VERY LONG TIMESCALES – LONGER THAN AGE OF UNIVERSE

B25 JEANS INSTABILITY + JEANS LENGTH

- IMAGINE A SMALL PERTURBATION IN A HOMOGENEUS MEDIUM (ρ_0)
- PREASSURE MAY NOT BALANCE GRAVITY, PERTURBATION COLLAPSES

① PREASSURE

- COMPRESS VOLUME $V \to V' = V(1-\epsilon)$ $0 < \epsilon \ll 1$
- PREASSURE INCREASES $p \to p' = \frac{\partial p}{\partial \rho} \cdot \rho'$ PER MASS $m = \frac{4\pi}{3} \rho_0 \cdot R^3$

$$\to F = \frac{\rho'}{\rho_0 \cdot R} = \epsilon \cdot \frac{c^2}{R}$$

② GRAVITY $\cdot F \simeq G \cdot \rho_0 \cdot R \cdot \epsilon$

→ GRAVITY WINS IF $\epsilon G \rho_0 \cdot R \gtrsim \epsilon \frac{c^2}{R}$

→ DEFINES SCALE $R = \frac{c}{\sqrt{G \cdot \rho_0}}$ WITH $c^2 = \frac{\partial p}{\partial \rho}\Big|_{\rho_0}$

OR COMPARE DYNAMICAL TIME SCALE $t_{DYN} = \frac{1}{\sqrt{G \cdot \rho_0}}$ WITH SOUND CROSSING $t_{SOUND} = \frac{R}{c}$ ON WHICH PREASSURE EQULIBRIUM TAKES PLACE.

JEANS INSTABILITY: ON SCALES $> R$ GRAVITY WINS AND THE SYSTEM COLLAPSES.

JEANS LENGTH

- LET'S LOOK AT THE FLUID EQUATIONS

$$\frac{\partial}{\partial t} \rho + \rho_0 \operatorname{div} \vec{v} = 0 \quad \Big| \frac{\partial}{\partial t} \qquad \text{CONTINUITY EQ.}$$

$$\frac{\partial}{\partial t} \vec{v} = -\frac{\nabla p}{\rho_0} - \nabla \phi \quad \Big| \operatorname{div} \vec{v} \qquad \text{EULER EQUATION}$$

$$\Delta \phi = 4\pi G \cdot \rho \qquad \text{POISSON EQUATION}$$

$$\rightarrow \underbrace{\frac{\partial^2}{\partial t^2}\rho - c^2 \Delta \rho}_{\text{SOUND WAVE}} = \underbrace{4\pi G \rho_0 \cdot \rho}_{\text{GRAVITY}} \qquad c^2 = \frac{\partial p}{\partial \rho}\Big|_{\rho_0}$$

LET'S TRY OUT A WAVE SOLUTION $\rho' = c \cdot e^{(i \cdot (kx - \omega t))}$

$\rightarrow$ DISPERSION RELATION

$$\omega^2 = c^2 k^2 - 4\pi G \cdot \rho_0$$

$$\frac{\omega}{c} = \sqrt{k^2 - \underbrace{\frac{4\pi G \cdot \rho_0}{c^2}}_{k_J^2}}$$

- STABILITY DEPENDS ON WAVE NUMBER $k_J = \sqrt{\frac{4\pi G \cdot \rho_0}{c^2}}$

if $k > k_J$: $\omega(k)$ IS REAL VALUE, WAVES (OSCILLATIONS)

$k < k_J$: $\omega(k)$ IS IMAGINARY, GRAVITATIONAL COLLAPSE, UNSTABLE SYSTEM

JEANS LENGTH $\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi}{G \cdot \rho_0}} \cdot c$